

# THE MATHEMATICAL GAZETTE

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*'I hold every man a debtor to his profession, from the  
which as men of course do seek to receive countenance  
and profit, so ought they of duty to endeavour themselves  
by way of amends to be a help and an ornament there-  
unto.'*

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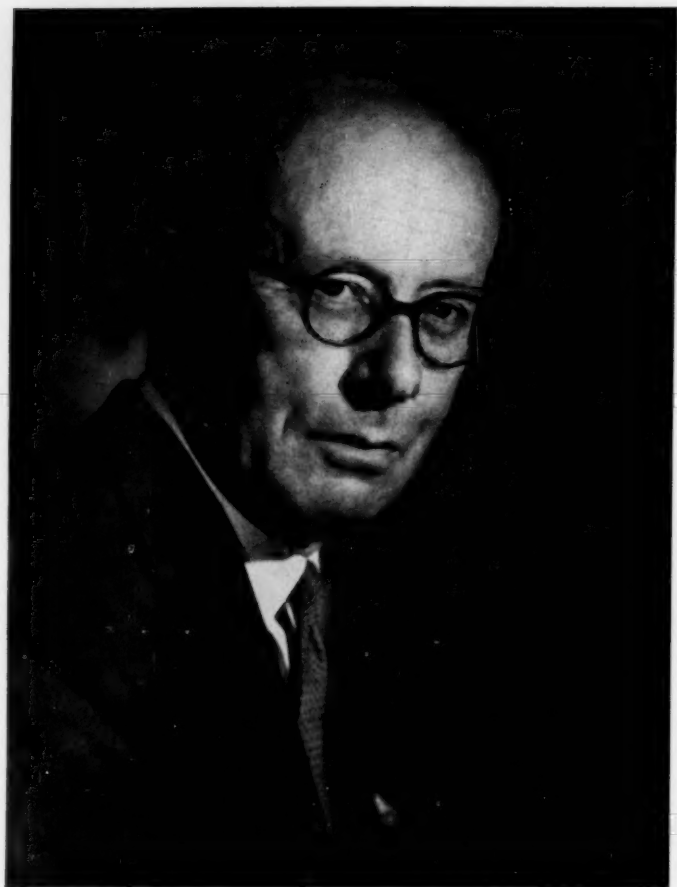
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Professor M. H. A. NEWMAN, F.R.S.  
*President, 1958-1959*

THE  
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WHAT IS MATHEMATICS?

NEW ANSWERS TO AN OLD QUESTION

*Presidential Address to the Mathematical Association*

April 1959

BY M. H. A. NEWMAN

The title of my address is the question with which Professor Temple ended his presidential address two years ago, when he gave such a fascinating review of the impulses and influences, both external and internal, that have promoted the growth of mathematics. Whether he expected an answer I cannot tell; but this Association, since its foundation in 1897, has frequently taken as the theme of its discussions and reports the duty of teachers to show the meaning and inner unity of the subject, and not merely to teach techniques. So it is not out of place to consider from time to time whether we are agreed what this meaning is, and what the techniques have to do with it.

The present is a particularly suitable time for such a review since there has rarely been a time when the work of mathematicians has been so strongly influenced and directed by their views on the nature of the subject. From the time of the revival of mathematics in the sixteenth century until well on into the nineteenth, mathematicians, although they took an interest in logical criticisms of the foundations, made mainly by people outside the fold, did not allow themselves to be deflected by such criticisms from doing what they pleased. Numbers which seemed to be imaginary or impossible, consecutive points, divergent sums, and so on, were used without hesitation and with gusto. Indeed, in the first great rush forward after the discovery of the calculus and infinite series, philosophic doubts could have done nothing but harm. The sudden end of the golden age of Greek mathematics after the death of Archimedes

may perhaps have been due partly to the crushing effect of the standard of rigour set by his works upon his less gifted followers. Fortunately by the time that these notions of Eudoxus and Archimedes were re-introduced by Dedekind 2000 years later, analysis was sufficiently advanced to assimilate them, and even to find in them a source of new ideas.

In the latter half of the nineteenth century there began a more penetrating analysis of the foundations of mathematics than there had ever been before. The traditional idea of mathematics, still accepted in the world at large to-day, and enshrined in the Oxford Dictionary's definition, is "the abstract science of space and number." The mathematical logicians of the late nineteenth century began to undermine the position that mathematics can be characterised in terms of the kinds of thing it deals with (space and number), by analysing them in terms of more primitive ideas. All the concepts that came into the mathematics of the day were shewn to be expressible in terms of the natural numbers, 0, 1, 2, ..., (geometry being arithmetised by means of coordinates); and the culmination of the movement was the famous definition, given independently by Frege and Russell, of the natural numbers themselves as classes of matching classes. Classes of *what* did not matter,—just all the classes that can be matched one to one with a given class.

By this final step it appeared that mathematics was reduced to a part of logic; indeed, the avowed intention of Whitehead and Russell in writing *Principia Mathematica* was to shew how the whole of mathematics could be deduced from a handful of initial "primitive propositions" of a purely logical kind.

In the long centuries of logical development that led up to this climax one thing that did not vary was the view that the objects occurring in mathematical theorems, whether they are geometrical figures and numbers or those classes or propositional functions in terms of which Frege and Russell sought to explain them, are permanent external realities, waiting to be discovered by mathematical explorers. Of this Platonic view of mathematical ideas G. H. Hardy gave one of the latest and best-known statements. "I believe," he said in *A Mathematician's Apology* (p. 63), "that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems we prove, and which we grandiloquently describe as our 'creations,' are simply our notes of our observations." According to Russell the mathematical propositions themselves exist independently of the various expressions of them, and existed before there were any languages to express them in.

This static view of mathematics led quite naturally to the emergence of the strange doctrine that mathematics consists of highly

complicated combinations of very simple primitive logical statements which, if you examine them closely, are found to be just tautologies,  $a = a$  or  $p$  implies  $p$ . Since a collection of tautologies, however large, is still a tautology, it follows that mathematics itself is just a vast tautology,—not an inspiring doctrine about the meaning of mathematics for teachers to pass on to their pupils. Although there was some attempt to say that “tautology” is here a technical term, and no reflection at all on the value of mathematics, the doctrine produced a feeling of disappointment in those who believed in it. In an essay, written on his eightieth birthday\* Russell says “I set out with a more or less religious belief in a Platonic world, in which mathematics shone with a beauty like the last cantos of the *Paradiso*. I came to the conclusion that the eternal world is trivial, and that mathematics is only the art of saying the same thing in different words.”

Soon after the beginning of this century two new views of the nature of mathematics began to be heard, which, though strongly opposed to each other, agreed in their rejection of the logical analysts' interpretation. About the “intuitionist” views of Brouwer I shall have little to say, because, although they have had a profound influence on our views about foundations, they have had little on the direction taken by contemporary mathematics itself. The views of Hilbert, on the other hand, have in fifty years led to a transformation of the whole aspect of mathematics. If one may attempt to summarise his view in a sentence it is that, whether or not mathematics is logic, mathematicians are certainly not logicians, and they should apply themselves not to such questions as what numbers and spaces really are, which belong to philosophy, but to constructing and developing formal methods of attacking problems. The mathematician wants to know not as much but as little as possible about the objects of his theories, provided that this little is precisely stated in the form of postulates or axioms, and suffices for the matter in hand.

To see how this attitude may have a direct effect on the development of mathematics itself, let us take the natural numbers as an example, and, to begin with, ask what are the properties that are needed to make them satisfactory for their most primitive use, to count things with. What does a shepherd do when he counts his flock? According to Frege and Russell, this can only be fully explained by a reference to the class of all the classes that match his flock. But what the shepherd does is to recite a rigmarole beginning “one, two, three,” in fact to tick off his sheep against a standard series which he has learned to extend indefinitely. What properties must this standard series have if it is to work properly?

\* *Portrait from Memory*, London.

First, there must be an initial number that we start with, which in the ordinary way we call 1; but mathematicians sometimes prefer to start with 0. It makes no difference: I will choose 0. Secondly, we must know how to go on indefinitely, that is, how to form the next number, the *successor*,  $x'$ , of any given number  $x$ ; and thirdly we must not repeat ourselves: different numbers must have different successors. If we observe that to say that 0 comes first is the same as to say that it is not the successor of any number, it will be seen that all three conditions are expressible in terms of the notion "successor of  $x$ ", or " $x'$ ." In fact, putting the second condition first, which is more natural, the three may be written so:

- (1) every number  $x$  has a unique successor,  $x'$ ;
- (2) there is a number 0, and  $0 \neq x'$  for all  $x$ ;
- (3) if  $x \neq y$  then  $x' \neq y'$ .

Finally if we are to do a little more than mere counting we need the principle of induction, which can be given the form

- (4) in every (non-empty) set of numbers there is at least one that is not the successor of any of the others.

Any set of objects with these properties will do very well as a standard series for counting and indeed for the whole of arithmetic. The figures in the scale of ten that we ordinarily use, consisting of rows of the digits 0 to 9, have all these properties if the successor is formed according to the rules that we learn at a tender age.

This is a very different way of characterising numbers from that of the Frege-Russell "logistic" school, and indeed it was strongly attacked by Russell, who said (in the *Introduction to Mathematical Philosophy*, chapter VII) that "the method of postulating what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil." This high indignation was quite misplaced. The purpose of axiomatic theories is not to make a philosophical investigation, but to provide the apparatus for a piece of work, in this case counting; and there is plenty of honest toil to be done. It must be shewn not merely that the numbers or other objects have the properties we want, but that this follows by strictly formal application of the axioms, in this case (1) to (4). This means that we are not to assume anything about the function  $x'$  except what is given in these four statements. In *choosing* the axioms we may use all the intuitive ideas we have about the "successor of  $x$ ," and consider too what use is to be made of the "numbers," whether by experts in the theory of numbers or by Bob the shepherd. But once the axioms are decided upon our proofs must use only what is stated in them and nothing more. This is the essence of an axiomatic

theory, that we are to abstain from using anything we happen to know about the objects of it that is not contained in the axioms.

But what can be the purpose of making things harder for ourselves by cutting ourselves off from part of what we know? It is quite a practical one. If the theorems really follow from the assumptions, and from them alone, they can be used whenever the assumptions are fulfilled, perhaps in quite different circumstances.

As an example let us consider what algebra makes of the idea of a *vector*. This object is first encountered as something which can be suitably represented by a line-segment with an arrow in it. But if we apply the criterion: "what properties do we need to use in dealing with vectors?" we find that there are really only two. First, any two vectors,  $a$  and  $b$ , have a *resultant*, which we call  $a + b$ ; secondly, if  $\lambda$  is a real number there is a vector  $\lambda a$ . The mere act of denoting the composition of vectors by "+" does not guarantee that it behaves like genuine addition; but in fact it does. It is commutative and associative, there is a zero vector, and every vector has its "equal and opposite" which when added to it gives the sum 0; and these are the four characteristic properties of addition. The multiplication of vectors by numbers also behaves as it should; for example  $\lambda(a + b) = \lambda a + \lambda b$ . Thus all we need to know about vectors for handling them in mathematics is that they can be added together, and multiplied by a number. To the axiomatic mathematician this suggests that it may be profitable to consider sets of things in general in which there are two operations, addition and multiplication by numbers, that obey the ordinary laws for these operations. As soon as we look around within mathematics we find a profusion of such sets. For example, all the polynomials in a variable  $t$  form such a set; and all the polynomials in  $t$  of degree  $\leq n$  form another, since the sum of two such polynomials, or the product of one of them by a number, is also a polynomial of degree  $\leq n$ . To all such sets we extend the name "vector space." Out of what started as a geometrical or even a physical notion there is distilled the axiomatically determined concept of a vector space, which may have as its elements such unexpected objects as polynomials, or infinite sequences, or even curves on a surface, if suitable methods are laid down of adding and multiplying by numbers.

The indifference of axiomatic theories to the nature of the objects they deal with could hardly be more clearly illustrated. It is clear that these are systems of a very different kind from such a deductive system as Euclid's *Elements*, where the initial assumptions are facts to be believed about the space we know.

The vector example shows also how apparently primitive ideas can be analysed into combinations of standard algebraical concepts, in this case a certain combination of addition with multiplication.

It is not only the concepts, but also theorems and proofs themselves that are subjected to this process of breaking into pieces, which are then the subjects of separate axiomatic treatment. This has been a particularly fruitful method in topology, where, for example, the theorem of Weierstrass that every bounded sequence of numbers has a convergent sequence has led on the one hand to the general notion of compact spaces, the most powerful source of topological existence theorems, and on the other hand to concepts of convergence to a point that can be applied to spaces in which the ordinary notion of a convergent sequence does not work properly. Topologists are indeed so far advanced in the art of breaking proofs into little pieces that it is now possible to take the greater part of a book to prove that every loop on a sphere can be filled in with a membrane lying on the sphere. Of course the quarry in such cases is not the theorem itself, which was quite well proved long ago, but by minute analyses and inspired generalisations of all the parts of some famous proof, to discover new methods which will yield a crop of theorems in new and unsuspected regions.

The introduction of highly organised axiomatic structures has brought extraordinarily rich rewards, in the shape of massive advances, for example, in algebraic geometry of any number of dimensions and in differential geometry, which were entirely beyond the reach of the mathematical methods of even twenty-five years ago.

I have conducted you into a country of abstractions which is far removed from the pleasant garden of numbers and spaces in which Hardy and H. F. Baker conducted their explorations. Theories in which intuition may have no place, about structures which are mere hypotheticals, things which *if* they have certain properties also have others, where all the parts of mathematics are made of the same materials and can be plugged in and out as required, —all this may seem a little uninviting, however splendid its results; perhaps a little too much like the world of Auden's recently published poem\* in which he speaks of

that ago when being was believing  
Truth was the most of many credibles,

in which, he says,

Truth was their model as they strove to build  
A world of lasting objects to believe in;

and contrasts it with

This while when, practical like paper-dishes  
Truth is convertible to kilowatts,

\* *In that ago*, by W. H. Auden, *Observer*, 29 March 1959.

Our last to do by is an anti-model  
An untruth anyone can give the lie to,  
A nothing no-one need believe is there.

That a mathematical theory is a lasting object to believe in few can doubt. Mathematical language is difficult but imperishable. I do not believe that any Greek scholar of to-day can understand the idiomatic undertones of Plato's dialogues, or the jokes of Aristophanes, as thoroughly as mathematicians can understand every shade of meaning in Archimedes' works. But that axiomatic theories are a "nothing no-one need believe is there" is a possibility that must be considered more seriously. We develop a theory about systems that obey certain axioms: but suppose there are no such systems, suppose even that after a great deal of work has been done, it is *proved* that there are none, by shewing that the axioms contradict each other in some subtle way that was not obvious to begin with? In one form or another this question has been the theme of the greater part of the work on the foundations of mathematics that has been done in this century. The most familiar way of giving a reassuring answer is to construct logical models of doubtful theories in more secure ones: complex numbers out of pairs of real ones, real numbers out of classes of rationals, rationals out of integers, and so on. But finally this backward or downward progress must come to an end in a basic theory with no deeper or surer theory beyond it to make a model in. How is *this* theory to be proved to be something rather than nothing?

This brings into view the controversies on the foundations that started fifty years ago and are still not resolved. The early successes of the axiomatic method led Hilbert to believe that mathematics is just the elaboration of such theories and nothing more. He took as his programme the complete mechanisation of mathematics by finding a general procedure for deciding the truth or falsity of mathematical theorems by examining the grammatical structure of the sentences they are expressed in. How these views were opposed by Brouwer and the intuitionists, and how finally in the 1930's the Hilbert reduction to grammar was shewn by Gödel to be inadequate, and the decision programme was shewn by Turing and Church to be impossible to carry out,—this is too big a subject to be touched upon incidentally. The effect of these discoveries of the thirties was not to destroy the value of axiomatic theories, but to bring them back to their proper place, as *methods* for attacking the problems of mathematics piecemeal; and methods are of course the heart of mathematics. If you are not interested in how equations are to be solved, or in all the details of a proof as well as in its conclusion, you are not interested in mathematics.

But there are other questions that can be asked about an axiomatic theory besides the fundamental one, whether it is something or nothing. It can be asked whether such remote abstractions can be either useful or interesting; and how can mathematicians work and make discoveries if they are to abstain from using any geometrical intuition? The answer is of course they don't abstain. The double life of mathematicians is a familiar fact to all who have been exposed to instruction in  $\epsilon$ -analysis. They know that diagrams and graphs are the life-line by which we survive amid rough seas of hard inequalities. If, to return to a previous example, we generalise the notion of a vector-space so far that it includes the set of all real polynomials in  $t$  as a special case, one of our purposes is to use some of our geometrical notions about the familiar vectors of statics and dynamics as a guide, as we grope our way towards the proof of a new theorem. When it is found, we shall carefully remove all traces of geometry or mechanics from our strictly axiomatic proof; and the man who reads it will probably put them in again as he grapples with the arguments.

The supreme task of mathematical analysis (using the word "analysis" in its proper and not its technical mathematical sense) is to be a source of new concepts, for science or for established parts of mathematics itself. During the last twenty or thirty years the view has been much in favour with scientists that a physical theory is nothing more than the aggregate of all the quantitative predictions about phenomena that can be derived from it. According to this view the possession of a computing machine which would print out the predictions when the conditions of the experiment are supplied to it would amount to having a physical theory even though nothing was known of what went on inside the machine. But people must think as well as calculate, and for thinking, especially about new and abstruse phenomena one needs new and perhaps rather abstruse *concepts*. Newton, besides giving us the dynamical equations of motion, gave us the concepts of linear and angular momentum which have enabled scientists and engineers to reason accurately and with assurance about the motion of material bodies; and these were mathematical concepts, new in their day. Classical analysis, in the technical sense, i.e. the theory of real and complex functions, has been a wonderfully fertile source of concepts for science, but it has recently begun to be plain that it is not quite flexible enough to handle all the ideas of physics. To express everything in terms of series and integrals imposes an artificial restraint which not only limits the range of available ideas, but has caused the rules and theorems to be hedged about with those elaborate conditions, congenial to analysts, which practical physicists find intolerable.

Now an axiomatic theory is exactly what is wanted to provide

concepts, made to measure, for a physical theory. The relations between physical entities that are proposed as laws, and these alone, can be taken as the axioms of a mathematical theory, and the theorems that follow shew what are the necessary consequences of the laws. For the working out of the details of particular examples the resources of classical analysis may be called upon, but the general axiomatic statement makes it possible to use different pieces of analysis for different problems, while keeping a unified set of concepts for all cases. The history of the Heaviside  $p$ -operator and the Dirac  $\delta$ -function provide striking examples. Heaviside's brilliant invention for dealing directly and simply with certain kinds of differential equations was first declared to be "not mathematics" by the mathematicians of the day when it appeared in the 1880's. Later it was provided with an elaborate function-theoretic interpretation by Bromwich and others, which still did not explain why Heaviside's simple but unorthodox methods happened to give the right answers. Only recently in Schwartz's axiomatic Theory of Distributions, and its variants by Temple and others, has a fully satisfactory account being given of the Heaviside operators and of the Dirac  $\delta$ , which had a rather similar career.

I was glad to see Heaviside's famous counterblast to his mathematical tormentors quoted, though with somewhat moderate approval, in the recent Ministry of Education pamphlet on teaching mathematics in secondary schools. His demand for a broader sort of logic, wider in its premisses, with more room for growth, has been met, without the loss of exactness which he thought was a necessary price to pay. Whether his other requirement, "more common-sensical," is satisfied by axiomatic mathematics in general is more doubtful; but physics itself has hardly gone that way.

It is time for me to come back to the questions which were my excuse for taking you so far afield. Have these generalities any bearing on the problem of showing pupils at school the true nature of mathematics? The underlying principle of axiomatic mathematics, that methods of doing things are what most interest mathematicians, or as I have put it, that methods are the heart of mathematics, can, I think have quite a direct influence on teaching methods. It implies that the meaning and interest of mathematics is to be sought in using it and seeing it in action, and that explanations can wait. This runs counter to the commonly accepted view that the reason for each rule of calculation must be made clear to the pupil before he starts to make systematic use of it, or even that the pupil should be led to discover the rules for himself. This has been strongly urged by Mr. Max Beberman in his Inglis Lecture at Harvard University.\*

\* *The Emerging Programme of Secondary School Mathematics*, by Max Beberman. Oxford 1959.

"Conventional text-books," he says, "place great stress on giving step-by-step algorithms for manipulation and simplification. Our contention is that these rules should be invented by students since they are merely short cuts in applying basic principles." It may be that if the students think hard enough and long enough, and are clever enough, they can discover all the rules; but since the deriving of short cuts from basic principles covers some of the finest achievements of the greatest mathematicians, I doubt it. It is mathematics itself that Mr. Beberman is sweeping aside with this phrase. He may, as he says, make school-children understand that a relation is just a class of ordered pairs, but if so he is teaching them mathematical logic, without first teaching them the mathematics that is its *raison d'être*. This seems an unnatural way to go to work. Mathematics, it has been said, is a language, a view that has much in common with what I have been saying; and it is a commonplace to-day that children should be taught to talk and read a little before they begin on the grammar. But "grammar" is a sad word to children, and I would rather regard mathematical theories as pieces of machinery to help us in our more exact thinking. This is not a comparison that we need shrink from if we remember that the machines are not automatic, but give plenty of scope for skill in playing on them; not automatic computers but the typewriter for punching instruction tapes.

How do boys, and men, become interested in machinery? By seeing it in action and learning to use it, and then, if they are of an inquiring turn of mind, finding out how it works,—not, surely, the other way round. It is true that if techniques are taught first, some time may elapse before the explanation comes; but I would not agree that these are periods of suspense, during which the work remains meaningless. Someone who knows the simple laws of electric currents in wires as facts knows something about electricity, even if he does not know how to deduce these laws from Maxwell's general equations of electromagnetism; and if a boy, in the course of learning how to use logarithms, gradually learns a coherent body of facts about them, he has learnt some mathematics, even before he finds out (which he will do with the more pleasure) that it all depends on indices. This, to return once more to my theme, is the essence of axiomatic mathematics: that the systematic development of the deductions from a set of abstract laws, and the study of the uses that can be made of them, make a branch of mathematics, even if they are so humble a set as the laws governing the use of logarithms, given as rules without reasons, in fact as axioms.

The authors of the Bourbaki treatise consider that you are not ready to hear about real numbers until you know about topological groups; so you see it is really all a matter of degree, of how far you

are prepared to push your principles. The pursuit of a final explanation will never succeed. It is better to recognise that each layer, or storey, has its own intrinsic value, greater or less; and as we ascend the sky-scraper of modern mathematics, in which, as Euclid observed, there are no elevators, the pleasure and enlightenment to be obtained is all the greater if we are thoroughly at home on one floor before starting to move to the next.

M. H. A. N.

## SUBTRACTION

BY ALICE S. BURSLEM

For some years I had been dissatisfied with the methods we were using in my small Primary School, to teach Subtraction. The slower children were unable to grasp the basic principles of Equal Addition, and many errors were made, as the children were using a "parrot" pattern, instead of basing their work on understanding and logical sequence.

When I analysed their mistakes, I discovered that most of them were caused by confusion in the mind of the child about the unit figure in the minuend. Both in Equal Addition and Decomposition this figure is added, in one method to the additional ten, and in the other, to the difference between the ten and the subtrahend unit number. I believed that if I could find a method in which only Subtraction was used, this difficulty would disappear. I also came to the conclusion that no method which allows any alteration to the subtrahend, either by adding or taking away, can be regarded as sensible. The subtrahend has no existence at all, as a separate entity. It is merely an integral part of the minuend. Thus it cannot, of itself, be added to, or taken from. It does not come into actual existence until the completion of the sum, when it becomes the number which, added to the answer, makes up the minuend. This will be very obvious to anyone who has tried to show in a practical way how Equal Addition is worked out.

I decided to see how the children themselves coped with the task of taking away a number of objects. I asked them not to count the number they were removing from the quantity, but to use the quickest method they could think of. For example, I gave these young Juniors six bundles of ten straws, and five odd straws. I asked them to place aside thirty seven of these straws, without untying more than one bundle. In almost every instance, the child put aside the five odd straws, took two more straws from one of the

tens, leaving eight loose ones; then three of the tens were put by these seven. They then counted the remaining ones, as, "I have two tens left, and eight loose ones ... twenty-eight." I tried the same kind of approach with other small objects, and I found each time that the child used all the objects equivalent to the units, and then made up the deficiency from the next ten. The remainder from this ten became the unit in the answer. The children soon grasped the fact that they must break into one ten, to make up the number they required. When I gave them shillings and pence, with one of the shillings made up of coppers, they used exactly the same method. A child who was given four shillings and four-pence, and put aside two shillings and five-pence out of this, always put the four-pence aside first, took one penny from the shilling which consisted of coppers, and then took the two shillings.

I let the children do a great deal of practical work before we worked any sums on paper. They had already worked, on paper, Subtraction sums in which the units number in the minuend was larger than that in the subtrahend, in which there is no difficulty. When the sums were now put down in a formal way on paper, I found that the method of approach used in practice was so well understood that even the slowest child had no difficulty in working out the answers. The "patter" of the beginner would be something like this. "I have seventy-six ... take away forty-nine. Give him the six, and another three from the ten, and he's got his nine. I have seven left out of that ten ... I put seven in the units' line. I give him the four tens he wants, from my six tens ... I used one of the seven tens so I only have six. When I've given him two, I have two tens left. Two in the tens line. I have twenty-seven left."

There is no addition, no mythical ten. This method appears to be perfectly logical to the child. He is not confused by processes which he cannot understand. He merely does in his mind what he has already done many times in real life ... used the part, before starting a new whole. He knows that he must first use the already opened tin of powder-paint, box of pen-nibs, jar of gum, etc., before he opens a new one. If he is required to give to each of his companions a new exercise book, he must first give out the few from the last packet which was opened, before he opens a new one, and makes up the deficit from that. He sees the grocer meticulously emptying the last grain of rice from the canister before he makes up the required lb from another supply. His mother half-fills the milk-jug from the bottle with a small quantity in it, before she opens the next pint of milk. Thus in his sum, he takes away the quantity, or sum of money, or unit number which he has, and makes up the rest from the next quantity, or shilling, or ten. After doing practical work, writing down the remaining numbers after given ones have

been taken away, the child becomes very quick at doing the mere formal sums which he sets down in his book. The reasoning is the same. It is based on common sense.

This method is even more useful in the type of sum when larger numbers are introduced, particularly in Weights and Measures. In a sum involving lbs, and ounces, whereas with Equal Addition, sixteen ounces would be added to the existing number of ounces, before the luckless child began to subtract, say, thirteen from it, ... in my method there is no addition at all. The child takes the number of ounces he has from the number to be subtracted, and the deficit is taken from the sixteen ounces. There are no complicated processes to baffle him.

The question may arise, how does the child cope with the position when there is no contribution from the next place ... such as five hundred and sixty seven to be taken from seven hundred and ninety nine. I find that no more confusion arises than when other methods are used, and probably less. I have frequently known children to add tens when no tens were needed, when I was teaching Equal Addition. Normally, however, the child realises that he can supply the given amount from his own supply.

I was interested to find that the young Junior identifies himself with the owner of the minuend, which is perfectly logical. Why else should he be interested in the remaining amount, after some of it has been subtracted? If I loan some of my money to a friend, surely I am the one who cares about my remaining cash, in which the borrower has no share. The shopper does not take an interest in how much produce the grocer has, after he has sold her the amount she requires; whereas it is a matter of concern to the shopkeeper. The pretended "he" was introduced at an early stage, to account for the situation in which a given quantity must be taken from the primary quantity or amount.

My own experience proves that the children work all kinds of subtraction sums much more quickly, with far greater accuracy, and with real understanding when this method is used.

*Geldeston Primary School,  
Norfolk*

*ALICE S. BURSLEM*

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### GLEANNING FAR AND NEAR

1928.  $\pi = \frac{1}{2}\pi$ .—His replies may be regarded as both complementary and supplementary to the picture of the present position in the disarmament talks seen through Western eyes.—*Manchester Guardian*, 20 July 1957. [Per Mr. R. F. Wheeler.]

## THE SCIENCE OF MECHANICS

By A. W. BELL

Perhaps the most notable achievement of the Mathematical Association since its formation has been the abolition of Euclid. While Euclid remains the greatest mathematical classic, it is no longer used as an instrument of torture on the young. As a school text-book Euclid is dead, replaced by a course in the well-established three stages, in which the type of learning and the degree of logical rigour are adapted to the understanding of the pupil.

But the ghost of Euclid has risen in a new guise, and his chilling breath of excessive rigour is now affecting the teaching of mechanics.

There is a difference of approach between the mathematical physicist and the applied mathematician. The latter starts with the minimum of postulates, and builds up a system in which the later theorems can be verified by reference to the natural world. He is interested primarily in producing a logically consistent system. The physicist is freer with his assumptions, less concerned with the elegance of his theory, primarily interested in giving an accurate description of nature in terms of concepts which appeal to the imagination. For a first course in mechanics, the physical approach is the right one; a second course may be more logical. The older text books do use the physical approach. They appeal widely to practical experience, and are not afraid to base their laws on experiment rather than deduction (e.g. the lever and the parallelogram of forces). This has two advantages; besides being easier, it focuses attention on the mechanical principles themselves rather than on the logical relations between them.

Mechanics is a strange and difficult subject when first encountered. Force, mass, acceleration, energy are abstract concepts which are not easy to understand. Since they are the bricks with which the structure of the science is built, it is important that the early stages of the subject should introduce them carefully, and should concentrate on developing the understanding of them and of the laws connecting them. This is stage A of mechanics, corresponding exactly with the stage A of geometry. The study of mechanical principles in their logical sequence belongs to stage C, the systematising stage, which comes in about the third year of the study of the subject. The modern mechanical Euclids, treating the subject from the point of view of the mathematician, introduce difficult concepts in too abstract a way, and deduce important principles in a way which is logically correct, but whose subtlety eludes the pupil. No wonder he emerges from the struggle clutching the formula  $P = ma$ , and applying it blindly to any numbers which come to hand.

I suggest, then, that we should take over from geometry this fruitful idea of the three stages, and conceive the school mechanics course as follows:

1st year: Stage A:— Introduction of all the basic concepts, including energy and momentum. Free use should be made of experience, leading to experiment; this should override the desire to develop the subject deductively. The examples should be mathematically simple, and should exhibit the use of these concepts in a wide variety of practical situations. There should be plenty of qualitative examples, too, of the "bus round corner" type. This year's work might well lead up to the O level Additional Mathematics examination. The mechanics questions in this examination ought to be framed so as to demand mechanical insight, and not mathematical ingenuity, following, at a lower level, the style of the Cambridge Mechanical Sciences Qualifying Exam.

2nd year: Stage B: Extension of the principles to more difficult applications. The approach becomes more deductive and less experimental, but remains practical, based on the study of concrete problems rather than types of motion, e.g. weight swung on a string, rod pivoted.

3rd year: Stage C: In this year's work, which includes virtual work and most of the rigid dynamics, the previous knowledge is systematised, and the deductive method brought to the fore. The year should begin with a review of all the earlier work, with particular attention to its logical foundations; there should be a discussion of Newton's laws, and of the evidence on which they rest; the distinction between gravitational and inertial mass may be drawn. There may be discussion of the nature of concepts and of the way in which the scientific method uses them. All this, and in fact the whole course, forms an illustration of the nature of science and the scientific method, and of the way in which a particular science develops.

A course arranged in these stages would, I believe, not only be educationally sound, but would reflect more truly the nature of the subject which should be not "Applied Mathematics"—a slave-subject providing convenient material on which to set mathematical problems—but "The Science of Mechanics."

*Mill Hill School*

A. W. B.

# A GENESIS FOR BINOMIAL IDENTITIES

BY ALBERT WILANSKY

**THEOREM A.** Let  $g$  be an even or odd function defined on  $(-1, 1)$ . Let  $s_n = \int_0^1 t^n g(2t-1) dt$ . Then  $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k = \pm s_n$ , + if  $g$  is even, - if  $g$  is odd.

Example 1.  $g(t) = 1$ ,  $s_n = \frac{1}{n+1}$ , resulting identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+1} = \frac{1}{n+1}.$$

Example 2.  $g(t) = t$ ,  $s_n = \frac{n}{(n+1)(n+2)}$ , resulting identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{k}{(k+1)(k+2)} = -\frac{n}{(n+1)(n+2)}.$$

Example 3.  $g(t) = \frac{1}{2} - \frac{1}{2}t$ ,  $s_n = \frac{1}{(n+1)(n+2)}$ . Here  $g$  is neither even nor odd. Let  $g_1(t) = \frac{1}{2}$ ,  $g_2(t) = -\frac{1}{2}t$ ,  $s_n^{(1)} = \int_0^1 t^n g_1(2t-1) dt = \frac{1}{2n+2}$ ,  $s_n^{(2)} = -\frac{n}{(2n+2)(n+2)}$  (check:  $s_n = s_n^{(1)} + s_n^{(2)}$ ), and the resulting identity is

$$\sum (-1)^k \binom{n}{k} \frac{1}{(k+1)(k+2)} = s_n^{(1)} - s_n^{(2)} = \frac{1}{n+2}.$$

**Proof of Theorem A.** Let  $f$  be defined by  $f(t) = g(2t-1)$ ,  $0 < t < 1$ . Then  $s_k = \int_0^1 t^k f(t) dt$ , and  $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k =$

$$\begin{aligned} \sum (-1)^k \binom{n}{k} \int_0^1 t^k f(t) dt &= \int_0^1 f(t) \sum (-1)^k \binom{n}{k} t^k dt = \\ \int_0^1 f(t)(1-t)^n dt &= \int_0^1 f(1-t)t^n dt = \int_0^1 t^n g(1-2t) dt = \\ \pm \int_0^1 t^n g(2t-1) dt &= \pm s_n \text{ according as } g \text{ is even or odd.} \end{aligned}$$

**THEOREM B.** Let  $g$  be a function defined on  $(-1, 1)$ . Let  $s_n$  be as in Theorem A. Then  $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k = s_n^{(1)} - s_n^{(2)}$  where  $s_n^{(i)} = \int_0^1 t^n g_i(2t-1) dt$ ,  $i = 1, 2$ ;  $g = g_1 + g_2$ ,  $g_1$  even,  $g_2$  odd.

The proof is clear (see Example 3).

It is possible to give a more useful form of Theorem B. We note that, for example, if  $\int_0^1 |f(t)| dt < \infty$  it follows that  $\int_0^1 t^n f(t) dt \rightarrow 0$  as  $n \rightarrow \infty$ . Thus we could not obviously apply Theorem B to  $s_n$  if  $s_n \rightarrow 0$ . In the next result we remove this restriction by using a Stieltjes integral.

**THEOREM C.** *Let  $h$  be an even or odd function defined on  $[-1, 1]$ , let  $q(t) = h(2t - 1)$ . Let  $s_n = \int_0^1 t^n dq$ . Then  $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k = \pm s_n$ , — if  $h$  is even, + if  $h$  is odd.*

Note: the signs are now reversed as compared with the earlier theorems.

The proof is the same as that of Theorem A.

A result similar to that of Theorem B is now obviously true in which Theorem C is modified by dropping the even-odd assumption.

**Example 4.**  $h(t) = -1, -1 \leq t < 0$   
 $1, 0 < t \leq 1$ .

Then  $q(t) = -\frac{1}{2}, 0 < t < \frac{1}{2}$   
 $\frac{1}{2}, \frac{1}{2} < t < 1$ ,

$s_n = 2^{-n}$ . The resulting identity is well known.

The following alternative form to Theorem B was suggested by R. L. Goodstein.

**THEOREM D.** *Let  $g$  be defined on  $(-1, 1)$  and let*

$$s_n = \int_0^1 t^n g(t^2 - t) dt$$

then 
$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_k = s_n.$$

The proof is the same as that for Theorem A.

*Lehigh University, Bethlehem, Pennsylvania*

A. W.

**1929.** I am by no means trying in this work to prove that the chimpanzee is a marvel of intelligence (one is obliged today to mention in a serious book that the chimpanzee has up till now shown no inclination or gift, for instance, for the study of fourth roots or elliptic functions)—Wolfgang Köhler "The Mentality of Apes" Pelican ed. 1957 page 176. [Per Mr. C. C. H. Barker.]

# CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

We wish to draw attention to some inaccuracies in the table of frequency of the digits of  $\pi$  submitted by the boys of Reading School.

The 15th row is completely wrong, and there are minor slips in rows 12 and 16.

The amended table below was compiled by two groups working independently, and where it differs from the Reading table the figures have again been double checked. We have been in correspondence with Mr. Vieri, who agrees that our figures are correct, and asks us to notify you.

Digits	0	1	2	3	4	5	6	7	8	9	Cross totals
	45	59	54	51	53	50	48	36	53	52	501
	48	57	49	52	40	47	46	59	48	54	500
	50	47	50	44	47	59	48	48	48	59	500
	39	49	54	42	55	49	58	54	53	47	500
	34	35	56	36	60	63	58	41	57	60	500
	43	62	40	41	63	47	44	49	51	60	500
	58	61	52	44	40	55	50	51	42	47	500
	45	59	53	59	47	47	46	39	53	52	500
	52	49	37	42	50	68	64	55	40	43	500
	52	54	51	49	53	40	51	56	47	47	500
	41	49	50	56	49	45	61	58	43	48	500
	50	45	48	57	56	52	45	60	47	40	500
	53	56	48	53	48	48	43	48	47	56	500
	47	51	50	61	41	60	46	40	51	53	500
	43	49	48	56	58	60	52	51	39	44	500
	59	44	47	56	55	45	49	58	42	45	500
	55	52	44	61	51	47	54	48	49	39	500
	59	39	64	51	45	44	69	39	45	45	500
	49	41	50	49	56	50	45	51	49	60	500
	39	50	55	41	44	55	49	59	49	59	500
Totals	961	1008	1000	1001	1011	1031	1026	1000	953	1010	10,001

Yours etc., PATRICIA CURPHEY  
MERLE D. KELLY  
PAT MOFFAT

For Lower Sixth Form,  
Douglas High School for Girls

To the Editor of the *Mathematical Gazette*

DEAR SIR,

It would be interesting to have a large number of terms of the expression for  $\pi$  as a continued fraction with unit numerators, which starts

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\dots}}}}}}}$$

For a random irrational number (assuming that 'random' can be defined!) the probability that a denominator of the continued fraction is a given positive integer  $r$  is easily shown to be  $1/(r+1)$ . Now some irrational numbers are clearly not random. For example, quadratic irrationals recur, and  $e$  has a regular pattern. A sufficiently long expression for  $\pi$  would indicate whether  $\pi$  is random in this sense.

Of course, a continued fraction has the advantage over a decimal that it is independent of the scale of notation.

Yours etc., E. J. F. PRIMROSE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

I would like to reply to one of the points raised by Mr. E. H. Lockwood (*Math. Gaz.*, 1958, **42**, 202). As a mathematician, he suggests that we should "teach our pupils to use letters to represent numbers, rather than distances, times or sums of money." As a teacher of chemistry I, along with many other teachers of the physical sciences, among whom I cite, in particular, Professor E. A. Guggenheim, instruct students in what Professor Guggenheim aptly terms the *quantity calculus* (*Journal of Chemical Education*, 1958, **35**, 606). It appears that the quantity calculus originated in the writings of A. Lodge (*Nature*, 1888, **38**, 281) and J. B. Henderson (*Math. Gaz.*, 1924, **12**, 99). In this calculus each letter, like  $P$ , symbolizes a physical quantity which is represented as the product of a measure (a real number) and an expression (often abbreviated) of the physical units which are being used. There are thus many possible representations of a physical quantity, as in the example  $P = 1 \text{ atm} = 1,013,250 \text{ dynes cm}^{-2} = 1.013250 \text{ bar} = 1.0332275 \text{ kg. cm}^{-2} = 76 \text{ cm mercury} = 29.92120 \text{ in. mercury} = 14.696006 \text{ lb. in}^{-2}$ . To illustrate I will translate into the language of the quantity calculus the following statement of Mr. Lockwood (*loc. cit.*). "At  $h$  feet above sea level the distance of the horizon is approximately  $\sqrt{3h/2}$  miles." In terms of the quantity calculus this becomes: "If  $h$  and  $d$  are the distances above sea level and to the horizon, then  $d/\text{miles} \simeq \sqrt{3h/2 \text{ feet}}$ ." The student of physical science eventually encounters

much more involved formulae than this. For instance, in many texts the Sackur-Tetrode entropy formula is written:

$$s = R\left(\frac{5}{2} \ln T + \frac{3}{2} \ln M - \ln P - 1.164\right), \quad (1)$$

and a note has to be added stating that  $M$  is the gram molecular mass of the gas concerned,  $T$  is the absolute temperature in degrees Kelvin ( $^{\circ}\text{K}$ ) and  $P$  is measured in atmospheres. In the quantity calculus equation (1) and this note are replaced by:

$$s = R\left(\frac{5}{2} \ln T/^{\circ}\text{K} + \frac{3}{2} \ln M/\text{g. mol}^{-1} - \ln P/\text{atm} - 1.164\right). \quad (2)$$

(In (1) and (2)  $R$  is the molar gas constant.) Another advantage of (2) is that it does not commit the error of taking logarithms of physical quantities (see G. N. Copley, *Journal of Chemical Education*, 1958, **35**, 366). Mathematicians insist, I understand, that this should never be done, and although I agree with them and have given, in the reference just cited, one instance of an inconsistency to which this leads, I would like to know of a more general mathematical discussion of the matter.

I hope, therefore, that teachers of mathematics will reconsider the teaching of the quantity calculus in dealing with applied mathematics. Am I right in saying that applied mathematics involves quantity calculus, whereas pure mathematics does not? I am sure that teachers of the physical sciences would welcome criticism of the quantity calculus, since it is most desirable that all teachers should be in agreement upon such an important subject.

Yours etc., G. N. COPLEY

To the Editor of the *Mathematical Gazette*

DEAR SIR,

The Pythagorean musical scale was abandoned in favour of equal temperament because the *intervals* of the latter, but not of the former, constitute a *group*. Does any reader know of any other conspicuous contributions made to life outside Mathematics by the elementary theory of groups? (I know about Campanology.)

Yours etc., A. W. FULLER

*Wanted*

Mathematical Questions from the Educational Times, 1912 to 1918 inclusive.

Journal of the Indian Mathematical Society, 1909 to 1933 inclusive.  
Lewent—Conformal Representation.

Hilton—Theory of Groups.

Ganguli—Theory of Plane Curves, Vol. I—3rd ed; Vol. II—2nd ed.

McDowell—Exercises in Euclid and Modern Geometry.

Payment in advance. Anyone having any of these to dispose of, may communicate with Clifford Marburger, Denver, Pennsylvania, U.S.A.

# MATHEMATICAL NOTES

## 2849. On a certain type of complex integral

The purpose of this note is to study the values of integrals of the type  $F[\gamma] = \int_{\gamma} \overline{g(z)} f'(z) dz$  where  $f(z)$  and  $g(z)$  are analytic in a connected region  $\Omega$ , and  $\gamma$  is any closed curve in  $\Omega$ . The notation  $\overline{g(z)}$  stands, as usual, for conjugate of  $g(z)$ .

We will prove that  $F[\gamma]$  is real for all  $\gamma \subset \Omega$ , if and only if the functions  $f(z)$  and  $g(z)$  are related by an identity of the form

$$g(z) \equiv aif(z) + m + ni \quad (1)$$

where  $a, m, n$ , are real numbers.

Similarly,  $F[\gamma]$  is purely imaginary if and only if

$$g(z) \equiv bf(z) + p + qi \quad (2)$$

holds for some suitable real numbers  $b, p, q$ .

Finally, after proving these two results, it is obvious that  $F[\gamma] \equiv 0$  if and only if at least one of the two functions  $f(z)$  and  $g(z)$  reduces to a constant.

*Proof:*

Excluding the cases  $f(z) \equiv \text{constant}$  or  $g(z) \equiv \text{constant}$ , we begin by proving that  $\Re F[\gamma] \equiv 0$  implies (1). (We use the notations  $\Re h$  and  $\Im h$  to denote the real and imaginary parts of any complex number  $h$ .)

Let be  $g(z) = u + iv$  and  $f(z) = \alpha + i\beta$ ;  $u, v, \alpha, \beta$  being functions of  $x$  and  $y$ . Then, if we write  $f'(z) = \alpha_x + i\beta_x$  with the subscripts denoting partial derivatives, the integral is

$$\begin{aligned} F[\gamma] &= \int_{\gamma} (u - iv)(\alpha_x + i\beta_x)(dx + idy) \\ &= \int_{\gamma} [(u\alpha_x + v\beta_x) dx - (u\beta_x - v\alpha_x) dy] + i \int_{\gamma} [(u\beta_x - v\alpha_x) dx \\ &\quad + (u\alpha_x + v\beta_x) dy], \end{aligned} \quad (3)$$

and the condition that  $\Re F[\gamma] \equiv 0$  for every  $\gamma \subset \Omega$ , implies

$$\frac{\partial}{\partial y} (u\beta_x - v\alpha_x) = \frac{\partial}{\partial x} (u\alpha_x + v\beta_x) \quad (4)$$

Now, if the Cauchy-Riemann equations for the functions  $g(z)$  and  $f'(z)$

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad \alpha_{xx} = \beta_{xy}, \quad \alpha_{xy} = -\beta_{xx} \quad (5)$$

are used to simplify the expansion of (4), we easily obtain

$$u_x \alpha_x + v_x \beta_x = 0$$

and, since the left-hand side of this equality is identical with

$$(\alpha_x^2 + \beta_x^2) \mathcal{R} \frac{u_x + iv_x}{\alpha_x + i\beta_x} = (\alpha_x^2 + \beta_x^2) \mathcal{R} \frac{g'(z)}{f'(z)}$$

we conclude that  $g'(z)/f'(z)$  is purely imaginary wherever  $f'(z) \neq 0$  and therefore, since  $f'(z)$  and  $g'(z)$  are analytic, we finally have

$$g'(z) = aif'(z) \quad \text{and} \quad g(z) = aif(z) + m + ni$$

with  $a$ ,  $m$  and  $n$  real constants.

Conversely, if (1) holds, we have

$$\overline{g(z)} = \overline{ai(\alpha + i\beta) + m + ni} = (m - a\beta) - i(n + a\alpha)$$

and by substitution of this value of  $\overline{g(z)}$ , together with  $f'(z) dz = (\alpha_x + i\beta_x)(dx + i dy)$  in  $F[\gamma]$ , we obtain, after easy manipulations, that the imaginary part of  $F[\gamma]$  is

$$\begin{aligned} \mathcal{I}F[\gamma] = \int_{\gamma} [\beta_x(m - a\beta) - \alpha_x(n + a\alpha)] dx \\ + [\alpha_x(m - a\beta) + \beta_x(n + a\alpha)] dy \end{aligned}$$

But this is the integral along the closed curve  $\gamma$  of an exact differential, because both terms in brackets are the partial derivatives of the function  $m\beta - n\alpha - a(\alpha^2 + \beta^2)/2$  as can be easily checked by using the Cauchy-Riemann conditions  $\alpha_x = \beta_y$  and  $\alpha_y = -\beta_x$ . Therefore,  $\mathcal{I}F[\gamma] = 0$  and  $F[\gamma]$  is real.

With trivial modifications, analogous arguments prove that the condition  $\mathcal{R}F[\gamma] \equiv 0$  is equivalent to  $\mathcal{I}[g'(z)/f'(z)] \equiv 0$  or to  $g(z) = bf(z) + p + qi$  with  $b$ ,  $p$  and  $q$  real numbers. The proof of the theorem is thus completed.

University of Massachusetts  
Amherst, Mass., U.S.A.

A. G. AZPEITIA

## 2850. On a conjecture of L. J. Mordell, II.

Let  $\mu(n)$  denote the greatest lower bound of

$$R_n(x_1, \dots, x_n) = \sum_{k=1}^{n-2} \frac{x_k}{x_{k+1} + x_{k+2}} + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2}$$

for  $x_k \geq 0$  ( $k = 1, \dots, n$ ).

L. J. Mordell [1] conjectured that  $\mu(n) < \frac{n}{2}$  for  $n \geq 7$ . It is now known (see [2] and [3]) that this inequality is indeed true for all even  $n \geq 14$ , and for all sufficiently large odd  $n$ . Moreover, R. A. Rankin [3] proved that

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\mu(n)}{n} = \inf_{n \geq 3} \frac{\mu(n)}{n} < \frac{1}{2} - 7 \times 10^{-8} < \frac{1}{2}.$$

Hence  $\mu(n) - \frac{n}{2}$  is not only negative for sufficiently large  $n$ , but tends to  $-\infty$  as  $n \rightarrow \infty$ .

Here we shall prove the following further results.

$$(2) \quad \lim_{n \rightarrow \infty} \frac{\mu(n)}{n} < \frac{1}{2} - 49,683 \times 10^{-8} = 0.49950317,$$

$$(3) \quad \mu(n) < \frac{n}{2} \text{ for all odd } n \geq 53.$$

Let

$$(h_1, \dots, h_{24}) = (7, 4, 6, 2, 5, 0, 3, 1, 1, 2, 0, 4, 0, 6, 0, 7, 1, 8, 3, 9, 5, 8, 6, 6),$$

$$b_{2k-1} = h_{2k-1}, \quad b_{2k} = 16 + h_{2k} \quad (k = 1, 2, \dots, 12).$$

Numerical calculation then shows that

$$R_{24}(b_1, \dots, b_{24}) < 12 - 0.011924,$$

and the result (2) follows from (1).

Now let us suppose that

$$R_N(a_1, \dots, a_N) = \frac{N}{2} - D, \text{ where } D > 0,$$

$$a_1 = 1 - \sigma, \quad a_2 = 1 + \sigma, \text{ where } \sigma > 0,$$

$$n = rN + 4m + 1, \text{ where } r \geq 1, m \geq 1,$$

and put

$$\delta = \frac{\sigma}{2m}, \quad U = \frac{1}{1 + \delta}, \quad V = \frac{1}{1 - \delta},$$

$$c_{2k-1} = 1 - 2(k-1)\delta, \quad c_{2k} = 1 + 2(k-1)\delta \quad (k = 1, 2, \dots, m+1),$$

$$a_{sN+k} = a_k \quad (s = 1, 2, \dots, r; k = 1, 2, \dots, N),$$

$$R_n = R_n(1, c_1, \dots, c_{2m}, a_1, a_2, \dots, a_{rN},$$

$$c_{2m+1}, c_{2m+2}, c_{2m-1}, c_{2m}, \dots, c_3, c_4).$$

Since

$$c_1 = c_2 = 1, \quad c_{2m+1} = a_1, \quad c_{2m+2} = a_2,$$

we obtain

$$\begin{aligned} R_n &= \frac{1}{2} + \sum_{k=1}^m \left\{ \frac{c_{2k-1}}{c_{2k} + c_{2k+1}} + \frac{c_{2k}}{c_{2k+1} + c_{2k+2}} \right\} + r \sum_{k=1}^N \frac{a_k}{a_{k+1} + a_{k+2}} \\ &\quad + \sum_{k=1}^m \left\{ \frac{c_{2k+1}}{c_{2k+2} + c_{2k-1}} + \frac{c_{2k+2}}{c_{2k-1} + c_{2k}} \right\} \\ &= \frac{1}{2} + r \left( \frac{N}{2} - D \right) + \sum_{k=1}^m \left\{ \frac{1 - 2(k-1)\delta}{2(1-\delta)} + \frac{1 + 2(k-1)\delta}{2} \right. \\ &\quad \left. + \frac{1 - 2k\delta}{2(1+\delta)} + \frac{1 + 2k\delta}{2} \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} + r \left( \frac{N}{2} - D \right) + \frac{2 + U + V}{2} m + (1 - V) \frac{m(m-1)}{2} \delta \\
&\quad + (1 - U) \frac{m(m+1)}{2} \delta \\
&= \frac{n}{2} - rD + \frac{U + V - 2}{2} (m - m^2 \delta) + \frac{V - U}{2} m \delta \\
&= \frac{n}{2} - rD + \frac{\sigma^2(4 - \sigma)}{8m} \frac{4m^2}{4m^2 - \sigma^2} < \frac{n}{2},
\end{aligned}$$

if  $r$  and  $m$  are large enough. Hence we can easily obtain a new proof that  $\frac{\mu(n)}{n} < C < \frac{1}{2}$  for all sufficiently large odd  $n$ .

To prove the result (3), put

$$N = 24, a_{2k-1} = \frac{h_{2k-1}}{11}, a_{2k} = 1 + \frac{h_{2k}}{11}.$$

Then

$$\sigma = \frac{4}{11}, D > 0.008895,$$

and if we choose  $r = 1, m = 7$ ,

$$n = 53, R_n < \frac{1}{2}n - 0.0003.$$

Hence  $\mu(53) < \frac{53}{2}$ , and the result (3) now follows from the inequality

$$\mu(n+2) \leq \mu(n) + 1$$

(see [2]).

References:

- [1] L. J. Mordell, On the inequality ... and some others, *Abhandl. Math. Seminar Hamburg* 22 (1958), 229-240.
- [2] A. Zulauf, Note on a conjecture of L. J. Mordell, *Abhandl. Math. Seminar Hamburg* 22 (1958), 240-241.
- [3] R. A. Rankin, An inequality, *Math. Gazette* XLII, No. 339 (1958), 39-40.

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A. ZULAUF

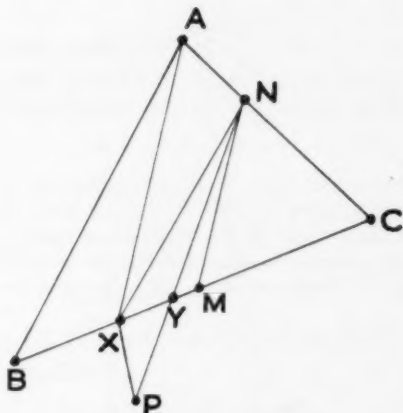
### 2851. Trials of a trisector

A powerful method for the solution of construction problems is that known as the method of "trial and error" or "false positions."

*Example:*—Given a triangle ABC and a point in its plane, required to draw through P a line which shall bisect the triangle.

For clarity, let us suppose the figure lettered so that  $P$  lies in the space bounded by  $BC$ ,  $AB$  produced, and  $AC$  produced.

Take any point  $X$  on  $BC$ , and through  $M$ , the middle point of  $BC$ , draw  $MN$  parallel to  $XA$  to meet  $AC$  in  $N$ . Join  $NP$ , and let it meet  $BC$  in  $Y$ .  $X$  and  $Y$  are connected by a rational (1, 1) relationship, and therefore different positions of  $X$  and  $Y$  constitute pairs of



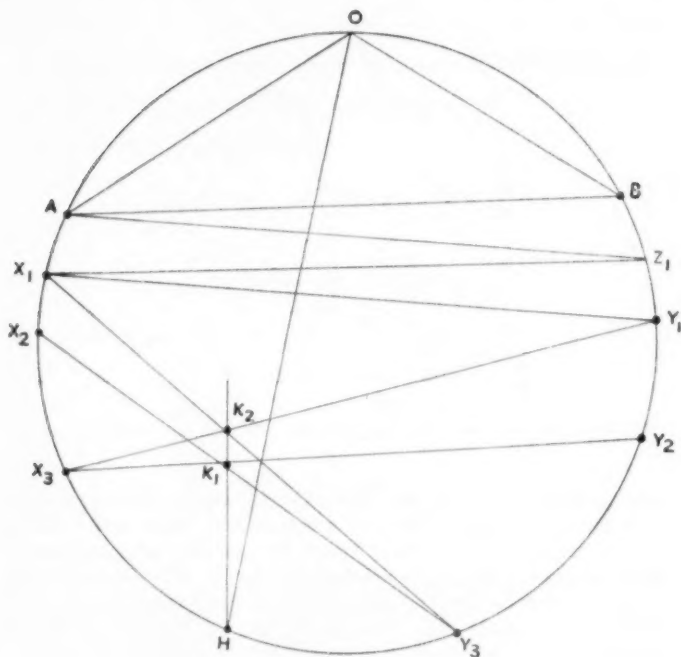
corresponding points of two homographic ranges. Construct three such pairs of points. There is a well-known ruler and compass construction for the common points of two homographic ranges, three pairs of corresponding points being known. Find these common points, and let the one that lies between  $B$  and  $C$  be named  $Z$ . Then  $PZ$  is the required line. For  $XN$  bisects the triangle,  $NYP$  are collinear, and  $Z$  has the properties of both  $X$  and  $Y$ .

The method must be used with great circumspection, as is shown by the following "trisection" of an angle.

Given an angle  $AOB$ ; required to trisect it.

Draw the circle  $OAB$ , and on the arc remote from  $O$  take a point  $X$ . (It will be convenient if  $X$  is fairly close to  $A$ .) Draw  $XZ$  parallel to  $AB$  to meet the circle again in  $Z$ . Join  $ZA$ . Draw  $XY$  parallel to  $AZ$  to meet the circle again in  $Y$ . Different positions of  $OX$ ,  $OY$  will be the corresponding rays of homographic pencils. Further, angle  $BOY$  is twice angle  $AOX$ , so that if a common ray of the two pencils could be found, it would be the required trisector.

Construct three positions  $X_1, X_2, X_3$  and  $Y_1, Y_2, Y_3$  of  $X$  and  $Y$ . Join  $X_2Y_3$  and  $X_3Y_2$  to meet at  $K_1$ . Join  $X_3Y_1$  and  $X_1Y_3$  to meet at  $K_2$ .  $K_1K_2$  will be the homographic axis of the pencils. Produce it to meet the circle, the point of intersection on the arc remote from  $O$  being  $H$ .  $OH$  will be the required common ray, and therefore the trisector.



The fallacy here is that, while any position of  $X$  yields a unique position of  $Y$ , every position of  $Y$  yields two positions of  $X$ . If we try to trace our steps back from  $Y$ , the first step, from  $Y$  to  $Z$ , could be described in various ways, but all would be tantamount to saying "bisect angle  $YOB$ ", and there are two bisectors. There is not a (1, 1) relationship between  $X$  and  $Y$ , and the pencils are not homographic.

In the valid example first cited, we can trace our steps back from

Y to X by the following constructions:—

Join PY and continue it to meet AC in N.

Draw AX parallel to NM to meet BC in X.

There is a (1, 1) relationship between X and Y, as can also be seen by having recourse to co-ordinate geometry.

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GEORGE TYSON

### 2852. When found, make a Note of

The jottings that compose this Note were made in the course of an attempt to find a geometrical setting for the familiar formula

$$R \sum \cos A = R + r. \quad (1)$$

1. If  $DEF$  is the pedal triangle of  $ABC$ , the triangle  $AEF$  has sides of lengths  $b \cos A$ ,  $c \cos A$  with an angle  $A$  between them; that is, it is a triangle similar to  $ABC$  with scale factor  $\cos A$ . Hence the perimeter of  $AEF$  is  $p(ABC) \cos A$ , where  $p(ABC)$  denotes the perimeter of  $ABC$ , and therefore, if  $ABC$  is acute-angled,

$$p(ABC) \sum \cos A = p(ABC) + p(DEF), \quad (2)$$

and (1) is equivalent to

$$p(ABC) : p(DEF) = R : r. \quad (3)$$

At first glance we seem to be no nearer our object, for the connection of  $r$  with  $DEF$  is remote, and so is the prospect of comparing the perimeters of two dissimilar triangles. But  $\frac{1}{2}r \cdot p(ABC)$  is  $\Delta$ , the area of  $ABC$ , and (3) is equivalent to

$$\frac{1}{2}R \cdot p(DEF) = \Delta. \quad (4)$$

Since  $OA$ ,  $OB$ ,  $OC$  are perpendicular to  $EF$ ,  $FD$ ,  $DE$ , it follows that if  $ABC$  is acute the products  $\frac{1}{2}R \cdot EF$ ,  $\frac{1}{2}R \cdot FD$ ,  $\frac{1}{2}R \cdot DE$  are the areas of the quadrilaterals  $OFAE$ ,  $OFBD$ ,  $ODCE$  which together make up the triangle  $ABC$ , and (4) is established.

It is not difficult to adapt this proof to the case in which the angle  $A$  is obtuse. The identity

$$\begin{aligned} (DF + FB + BD) + (ED + DC + CE) + (FE + EA + AF) \\ = (BC + CA + AB) + (ED + DF + FE) \end{aligned}$$

is still valid, but if  $BC$ ,  $CA$ ,  $AB$  are positive, then  $EA$ ,  $AF$ ,  $FE$  are negative, and instead of (2) we have, with positive lengths,

$$p(ABC)(\sum \cos A - 1) = ED + DF - EF.$$

Also the triangle  $ABC$  is now

$$(OCE - OCD) + (OFB - ODB) - (OFA + OAE),$$

with area  $\frac{1}{2}R(ED + DF - EF)$ , and the conclusion is reached as before.

This modification shews that to recognise  $FD + DE + EF$  as a perimeter really throws us off the track, unless we first transfer the whole enquiry to the geometry of direction, where the directions of the sides of the pedal triangle will be determined from those of the sides of the original triangle by the agreement that the lengths of  $EF$ ,  $FD$ ,  $DE$  are  $a \cos A$ ,  $b \cos B$ ,  $c \cos C$ , in sign as well as in magnitude.

2. The verification of formulae such as (1) is often facilitated by the use of the cubic equation whose roots are  $a$ ,  $b$ ,  $c$ . Since  $abc = 4R\Delta = 4sRr$ , the equation is

$$x^3 - 2sx^2 + qx - 4sRr = 0,$$

where  $q$  has yet to be found. Substituting  $s$  for  $x$  in the identity

$$x^3 - 2sx^2 + qx - 4sRr = (x - a)(x - b)(x - c)$$

we have

$$s(-s^2 + q - 4Rr) = \Delta^2/s = sr^2,$$

whence

$$q = s^2 + r^2 + 4Rr,$$

and symmetric functions of  $a$ ,  $b$ ,  $c$  can be evaluated in terms of  $s$ ,  $R$ ,  $r$  alone by the usual routine.

For example

$$\Sigma a^2 = 4s^2 - 2q = 2(s^2 - 4Rr - r^2),$$

$$\begin{aligned}\Sigma a^3 &= 2s \Sigma a^2 - q \Sigma a + 12sRr = 2s(4s^2 - 3q + 6Rr) \\ &= 2s(s^2 - 6Rr - 3r^2),\end{aligned}$$

$$\begin{aligned}2abc \Sigma \cos A &= \Sigma a(b^2 + c^2 - a^2) = \Sigma a \cdot \Sigma a^2 - 2\Sigma a^3 \\ &= 4s(s^2 - 4Rr - r^2) - (s^2 - 6Rr - 3r^2) \\ &= 8sr(R + r),\end{aligned}$$

and this, to end colloquially, is where we came in. E. H. NEVILLE

### 2853. Common root of two polynomial equations

Many algebra text-books give Sylvester's method of "dialytic elimination" or its equivalent to show that *the condition for equations*

$$f(x) \equiv a_0x^m + a_1x^{m-1} + \dots + a_m = 0 \quad (a_0 \neq 0)$$

$$g(x) \equiv b_0x^n + b_1x^{n-1} + \dots + b_n = 0 \quad (b_0 \neq 0)$$

to have at least one root in common is  $|\mathbf{R}| = 0$ , where  $|\mathbf{R}|$  is the determinant of the  $(m+n) \times (m+n)$  "bigradient" matrix

$$\mathbf{R} = \begin{pmatrix} a_0 & a_1 & \dots & a_m & & \\ & a_0 & a_1 & \dots & a_m & \\ & & \ddots & \ddots & \ddots & \\ & & & a_0 & a_1 & \dots & a_m \\ b_0 & b_1 & \dots & b_n & & \\ & b_0 & b_1 & \dots & b_n & \\ & & \ddots & \ddots & \ddots & \\ & & & b_0 & b_1 & \dots & b_n \end{pmatrix}$$

(all elements not written being 0).

The condition  $|\mathbf{R}| = 0$  thus established is only *necessary*; its sufficiency is seldom mentioned, but often assumed. The following proof that  $|\mathbf{R}| = 0$  is *sufficient* for  $f = 0$ ,  $g = 0$  to have a common root may therefore be of interest.

Write

$$y_r = f(x_r) \quad (r = 1, 2, \dots, n),$$

where  $x_1, \dots, x_n$  are the  $n$  roots (in complex algebra) of  $g(x) = 0$ . We require to prove that  $y_r$  is zero for at least one  $r$  ( $1 \leq r \leq n$ ). Let  $\mathbf{R}(y)$  denote the matrix obtained from  $\mathbf{R}$  by replacing  $a_m$  by  $a_m - y$  everywhere that it occurs.

The equations

$$g(x) = 0, \quad a_0 x^m + \dots + a_{m-1} x + (a_m - y_r) = 0$$

clearly have a common root  $x_r$ . Hence by the *necessary* condition applied to these equations,  $|\mathbf{R}(y_r)|$  will be zero. This holds for each  $r = 1, 2, \dots, n$ ; that is,  $y_1, \dots, y_n$  are the roots of  $|\mathbf{R}(y)| = 0$  (a polynomial equation in  $y$  of degree  $n$ ).

The hypothesis  $|\mathbf{R}| = 0$  is equivalent to  $|\mathbf{R}(0)| = 0$ , so that  $y = 0$  is a root of  $|\mathbf{R}(y)| = 0$ . Hence at least one of the  $y_r$  is zero.

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#### 2854. An approximation to $(a^2 + b)^{\frac{1}{2}}$

Since

$$(1 + 2x)^{\frac{1}{2}} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^4}{8} (1 + 2\theta x)^{-7/2}, \quad 0 < \theta < 1,$$

and

$$1 + x - \frac{1}{2} \frac{x^2}{1+x} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{1}{2} \frac{x^4}{1+x}$$

therefore the error in taking  $1 + x - \frac{1}{2} \frac{x^2}{1+x}$  for  $(1 + 2x)^{\frac{1}{2}}$  is

$$\frac{1}{3}x^4 \left\{ \frac{4}{1+x} - \frac{1}{(1+2\theta x)^{7/2}} \right\} = \frac{1}{3}x^4 R(x), \text{ say.}$$

If  $3 > x \geq 0$ ,  $0 < R(x) < 4$ , and if  $0 > x > -\frac{1}{3}$ ,  $0 < R(x) < 4$ .  
Hence

$$(a^2 + b)^{\frac{1}{2}} = a + \frac{b}{2a} - \frac{b^2}{4a(2a^2 + b)}$$

with an error less than  $b^{\frac{1}{2}}/32a^7$  when  $-\frac{1}{4} < \frac{b}{a^2} < b$ .

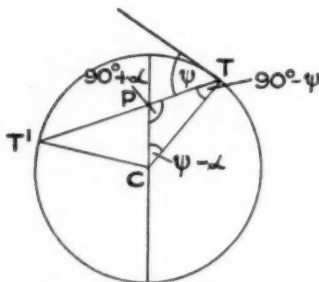
*Example.*  $\sqrt{26} = 5.099019513$  from tables, and by the above approximation is  $5 + \frac{1}{10} - \frac{1}{20.51} = 5.099019608$  with an error less than  $10^{-7}$ .

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### 2855. On note 2825; an alternative proof

Let  $T'PT$  be any chord through  $P$  and let  $C$  be the centre of the circle. From elementary considerations, if  $T'PT$  makes an acute angle  $\alpha$  (positive or negative) with the normal to the diameter both through  $P$ , the angles of triangle  $CPT$  are as shown in the figure



and from the sine rule

$$\frac{\sin(90^\circ - \psi)}{\sin(90^\circ + \alpha)} = \frac{CP}{CT}$$

which is a positive constant  $= k^2$ , say.  $\therefore \cos \psi = k^2 \cos \alpha$ .

If  $\psi$  is to be a minimum,  $\cos \psi$  must be a maximum, so that  $\cos \alpha$  must also be a maximum and therefore  $\alpha = 0$ .

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**2856. On the resultant of two quadratic forms**

If  $f_1 = 0$ ,  $f_2 = 1$  and, for all  $n \geq 1$ ,  $f_{n+2} = af_{n+1} + bf_n$  then the resultant of the quadratic forms

$$\begin{aligned} 1. & \quad f_{n+2}x^2 + 2f_{n+1}xy + f_ny^2 \\ 2. & \quad f_{n+k+2}x^2 + 2f_{n+k+1}xy + f_{n+k}y^2 \\ \text{is} & \quad b^{2n-2}(a^2 + 4b)f_{k+1}^2 \end{aligned}$$

We remark first that a simple induction over  $k$  establishes the formula

$$3. \quad f_{n+k} = f_n f_{k+2} + b f_{n-1} f_{k+1}, \quad n \geq 2.$$

If

$$\Delta_n = \begin{vmatrix} f_{n+2} & 2f_{n+1} & f_n & 0 \\ 0 & f_{n+2} & 2f_{n+1} & f_n \\ f_{n+1} & 2f_n & f_{n-1} & 0 \\ 0 & f_{n+1} & 2f_n & f_{n-1} \end{vmatrix}$$

then, subtracting  $a$  times the third from the first row, and  $a$  times the fourth row from the second row, we find  $\Delta_n = b^2 \Delta_{n-1}$ ; but  $\Delta_2 = a^2 + 4b$  and so  $\Delta_n = b^{2n-4}(a^2 + 4b)$ .

The resultant  $R$  of the forms 1 and 2 is

$$\begin{vmatrix} f_{n+k+2} & 2f_{n+k+1} & f_{n+k} & 0 \\ 0 & f_{n+k+2} & 2f_{n+k+1} & f_{n+k} \\ f_{n+2} & 2f_{n+1} & f_n & 0 \\ 0 & f_{n+2} & 2f_{n+1} & f_n \end{vmatrix}$$

Taking  $f_{k+2}$  times the third row from the first, and  $f_{k+2}$  times the fourth from the second we find

$$R = b^2 f_{k+1}^2 \Delta_n = b^{2n-2}(a^2 + 4b)f_{k+1}^2.$$

In particular if  $(f_n)$  is the Fibonacci sequence, so that  $a = b = 1$ , the resultant is  $5f_{k+1}^2$ .

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M. RUMNEY

**2857. From a script**

To prove  $\triangle ACP$  similar to  $\triangle BDP$

$$\widehat{PAC} = \widehat{PBD} \text{ (angles in the same segment)}$$

$$\widehat{DPC} = \widehat{DPC} \text{ (common sense).}$$

H. M. C.

**2858. A travelling triangle**

$A$  is the mid point of the arc  $BD$  of a circle of radius  $r$  and  $C, C'$  are the third vertices of the equilateral triangles described on  $AB$  as side.  $CD$  meets the circle again at  $E$ , and  $C'D$  meets the circle again at  $E'$ . Then  $CE = C'E' = r$  and  $BE = BE' = r$ .

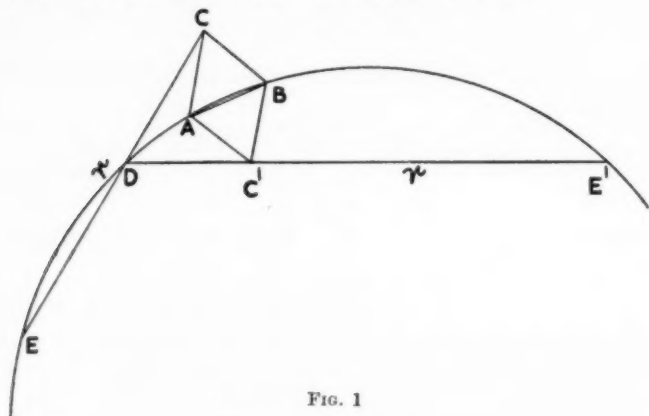


FIG. 1

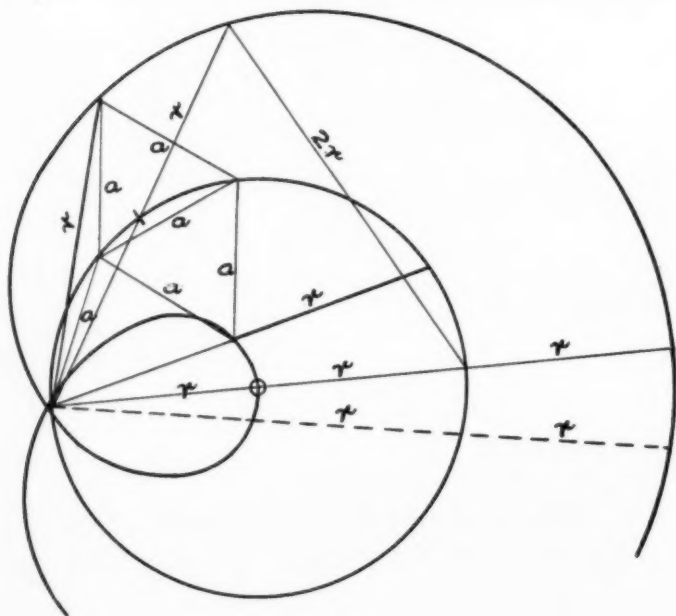


FIG. 2

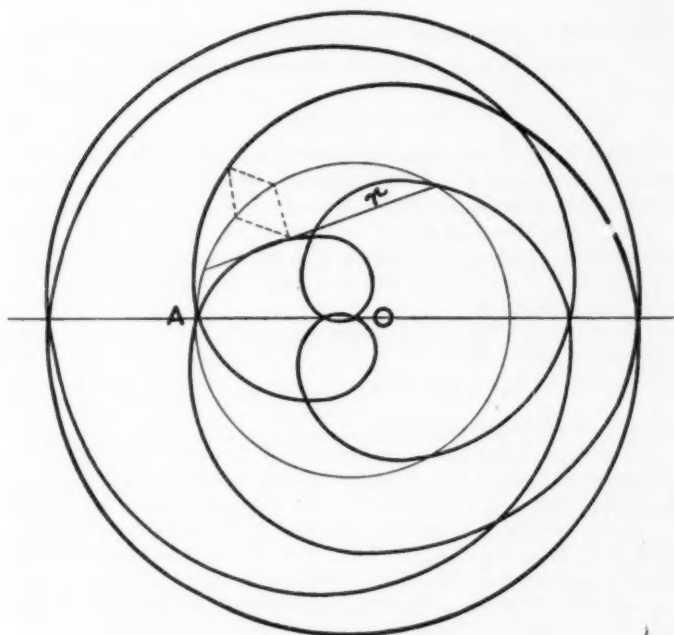


FIG. 3

Let  $O$  be the centre of the circle,  $\angle ODC' = \beta$ ,  $\angle ADC' = \alpha$ . Then  $\angle BAC = 2(\alpha + \beta) = 180 - 2\alpha + 60$  so that  $2\alpha + \beta = 120$  and therefore  $C'E' = 180 - \beta - (\alpha + 30) = \alpha + 30 = \angle OC'E'$ , so that  $C'E' = OE' = r$ , and similar arguments apply in the other cases. If the points  $A, B$  move round the circle keeping  $D$  fixed and  $DA = AB$  then  $C, C'$  describe a cardioid (Fig. 2). If arc  $DA = 2$  arc  $AB$  then the locus is the curve shown in Fig. 3.

ARCHIBALD H. FINLAY

### 2859. Maximising a determinant

1. *Notation.* We write  $F$  for any sum of 3 terms, each the product of 3 different elements. The expansion of a determinant gives two such expressions,  $F$ , one positive and one negative.

$x = (a, b, c)$  means that  $x = a$  or  $x = b$  or  $x = c$ .

$(x, y, z) = (a, b, c)$  means that  $x, y, z$  are respectively equal to  $a, b, c$  but not necessarily in that order.

$(a_1, a_2, a_3) > (b_1, b_2, b_3)$  means that every  $a$  is greater than every  $b$ .

2. LEMMA 1. If  $a, b, c, d$  are all positive and  $a > b, c > d$  then

$$(a - b)(c - d) = ac + bd - (ad + bc) > 0$$

so that

$$ac + bd > ad + bc \quad \dots (1)$$

LEMMA 2. If  $U$  and  $V$  are two terms in  $F$  and  $U \geq V$  and if  $U$  contains an element  $u$  and  $V$  contains an element  $v$  such that  $u < v$ , then interchanging  $u$  and  $v$  in  $F$  gives  $F_1 > F$ .

*Proof.* Let  $U = U_1u$  and  $V = V_1v$  (we may call  $U_1$  and  $V_1$  the co-factors of  $u$  and  $v$ ), then, since  $u < v$ , we must have  $U_1 > V_1$  and, by Lemma 1,  $U_1v + V_1u > U_1u + V_1v$  and the result follows.

3. THEOREM 1. If

$$F_r = a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 \quad \dots (2)$$

and

$$(a_1, a_2, a_3) > (b_1, b_2, b_3) > (c_1, c_2, c_3) \quad \dots (3)$$

Then  $F_r$  is the greatest possible  $F$ .

*Proof.* Let  $F$  have any arrangement of elements. We apply the operation of Lemma 2 to  $F$ , interchanging elements  $u$  and  $v$  as long as we can find  $u < v$  and  $U \geq V$ . By repeated application of this operation we obtain a series of  $F$ 's, each one greater than the last, thus  $F < F_1 < F_2 < F_3 \dots < F_r$ . The last  $F$  is reached when we cannot perform the operation any more because  $U > V$  and  $u > v$  for every  $U, V, u$  and  $v$ . In this case the inequalities (3) will hold, and the last, and greatest  $F$  must be  $F_r$  as in (2) above.

COROLLARY. If

$$(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) = (1, 2, 3, \dots, 9)$$

then it follows from the inequalities (3) that

$$(a_1, a_2, a_3) = (7, 8, 9), (b_1, b_2, b_3) = (4, 5, 6), (c_1, c_2, c_3) = (1, 2, 3) \quad \dots (4)$$

so that

$$F_r = 504 + 120 + 6 = 630$$

One possible arrangement for  $F_r$  is

$$F_r = 8.7.9 + 6.5.4 + 3.1.2$$

4. THEOREM 2. The next largest  $F$ , after  $F_r$ , is  $F_s = 602$ .

*Proof.* The proof of Theorem 1 shows that, given any  $F$ , we can construct a series  $F < F_1 < F_2 \dots < F_r$  of increasing value, in which each  $F$  differs from the next one by one interchange of two elements.  $F_s$  must be the penultimate term in such a series. It follows that  $F_s$  must differ from  $F_r$  by just one interchange of two elements. Let these elements be  $u$  and  $v$  with co-factors  $U_1$  and  $V_1$ , as in Lemma 2. Then

$$F_r - F_s = U_1u + V_1v - (U_1v + V_1u) = (u - v)(U_1 - V_1)$$

We have to find the smallest possible value for  $F_r - F_s$ . This can be done by a short calculation. We have only the following cases to consider:

	$u - v$	$U_1 - V_1$	$F_r - F_s$
These give the smallest possible values for $u - v$	$4 - 3 = 1$	$30 - 2 = 28$	28
	$7 - 6 = 1$	$72 - 20 = 52$	52
This gives the smallest possible value for $U_1 - V_1$	$6 - 1 = 5$	$20 - 6 = 14$	70

For any other values of the elements we have  $u - v > 1$  and  $U_1 - V_1 > 14$  so that  $(u - v)(U_1 - V_1) > 28$ . This proves that the smallest possible value for  $F_r - F_s$  is given by  $(4 - 3)(30 - 2) = 28$  and so  $F_s = 630 - 28 = 602$ .

5. THEOREM 3. The smallest possible value for  $F$  is  $F_a = 214$

*Proof.* Let  $F = U + V + W$ . Then  $UVW = 9!$ . Put  $G = F/3 - \sqrt[3]{UVW}$  then  $G \geq 0$  since the arithmetic mean is greater than, or equal to, the geometric mean. Also the minimum for  $G$  corresponds to the minimum for  $F$  since  $\sqrt[3]{UVW} = \sqrt[3]{9!} = 71.326$  is a constant. The minimum occurs when  $U = V = W = 71.326$ , so, since we are dealing with whole numbers, the minimum for  $F$  is greater than  $3 \times 71.326 = 213.97$ . In fact we find  $F_a = 1.8.9 + 2.5.7 + 3.4.6 = 214$ . This proves the Theorem.

6. THEOREM 4. If  $F_r$  has the form (2) of Theorem 1, and the inequalities (3) are satisfied, and we form a determinant the sum of whose positive terms is  $F_r$ , then the sum of its negative terms is not less than 218.

*Proof.* Let the determinant be

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 - (b_3a_2c_1 + c_3b_2a_1 + a_3c_2b_1) = F_r - F_b$$

where the  $a$ 's,  $b$ 's and  $c$ 's satisfy the inequalities (3) and the equations (4).

First it is clear that owing to the arrangement of elements in the terms of  $F_b$ ,  $F_b$  cannot be  $F_a$  and so  $F_b > 214$ .

Let  $F_b = hH + kK + jJ$  where  $h, k$  and  $j$  are elements and  $H, K$  and  $J$  are their co-factors, and where

$$(h, k, j) = (a_1, a_2, a_3) \text{ or } (b_1, b_2, b_3) \text{ or } (c_1, c_2, c_3) \\ = (7, 8, 9) \text{ or } (4, 5, 6) \text{ or } (1, 2, 3) \dots (5)$$

If we can find  $h, k, H, K$  such that  $h > k$  and  $H > K$  we can interchange  $h$  and  $k$  in  $F_b$  and get  $F_c = kH + hK + jJ$  and, by Lemma 1,

we have  $F_c < F_b$ . Note that  $F_c$  is also a possible sum of the negative terms of  $D$  owing to the relation (5). We can perform a similar operation, interchanging  $h$  and  $j$  or  $k$  and  $j$ . In order to satisfy the inequalities  $h > k$  and  $H > K$ , or similar inequalities for other letters, we may have to change  $h, k, j$ , always subject to the relation (5). We continue to carry out these operations as long as we can find elements  $h, k, j$  which satisfy the inequalities. But if

$$h < k < j \quad \text{and} \quad H > K > J \quad \dots (6)$$

for all possible values of  $h, k, j$  and  $H, K, J$ , then the inequalities cannot be satisfied and the operation is not possible. When this happens we shall have reached the smallest possible value for  $F_b$ . Now if the inequalities (6) hold for all values of  $h, k, j$  we can put  $h = 1, k = 2, j = 3$ , then  $H$ , by (6), must be as great as possible, i.e.  $H = 6 \times 9 = 54$ ,  $K$  must be the next greatest, i.e.  $K = 5 \times 8 = 40$  and  $J$  must have the smallest value, i.e.  $J = 4 \times 7 = 28$ . This gives  $F_c = 1 \times 54 + 2 \times 40 + 3 \times 28 = 218$ . One arrangement is  $F_c = 9.1.6 + 2.5.8 + 4.7.3 = 218$ .

A calculation shows that the inequalities (6) are satisfied for all possible values of  $h, k, j$ . We have

$$\begin{aligned} h, k, j: \quad & 1 < 2 < 3, \quad 4 < 5 < 6, \quad 7 < 8 < 9 \\ H, K, J: \quad & 54 > 40 > 28, \quad 21 > 16 > 9, \quad 12 > 10 > 6 \end{aligned}$$

Thus  $F_c$  must be the smallest possible value for  $F_b$  which proves the Theorem.

7. We have now shown that the greatest possible value for the determinant  $D = F_r - F_c$  is  $D^* = 630 - 218 = 412$ . Let  $D' = F_u - F_v$ , where  $F_u \neq F_r$ , be any other determinant. Then, by Theorem 1,  $F_u < 630$  and, by Theorem 2,  $F_u \leq 602$ . Also, by Theorem 3,  $F_c \geq 214$ . It follows that

$$D' \leq 602 - 214 = 388 < D^*$$

so that  $D^*$  is the greatest determinant.

8. If we put  $a_1 = 8, a_2 = 7, a_3 = 9, b_1 = 6, b_2 = 5, b_3 = 4, c_1 = 3, c_2 = 1, c_3 = 2$  we have the determinant

$$D = \begin{vmatrix} 8 & 6 & 3 \\ 1 & 7 & 5 \\ 4 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 8 \\ 7 & 2 & 6 \\ 5 & 9 & 3 \end{vmatrix} \quad \begin{array}{l} \text{by suitable changes of} \\ \text{rows and columns.} \end{array}$$

The second determinant above is Mr. Sinclair Grant's determinant, given in Mathematical Note No. 2792 in the "Mathematical Gazette" Vol. XLII, No. 342.

## 9. Examples of the operations in Theorems 1 and 4

(i) Example of the operations in the proof of Theorem 1. Elements to be interchanged are underlined.

$$F = 1.4.\underline{6} + \underline{2}.7.8 + 3.5.9 = 24 + 112 + 135 = 271$$

$$F_1 = 1.4.\underline{2} + \underline{6}.7.8 + 3.5.\underline{9} = 8 + 336 + 135 = 479$$

$$F_2 = 1.\underline{4}.2 + 9.7.8 + \underline{3}.5.6 = 8 + 504 + 90 = 602$$

$$F_r = 1.3.2 + 9.7.8 + 4.5.6 = 6 + 504 + 120 = 630$$

It may be noticed that we have always chosen  $u$  as the smallest element in  $U$ , and  $v$  as the greatest element in  $V$ . This is merely a matter of convenience. Other elements can be chosen, giving a different series, but the end result will always be the same.

(ii) Example of the operations of Theorem 4.

$$F_b = 4.\underline{9}.3 + 2.6.7 + \underline{8}.1.5 = 232$$

$$F_c = 4.8.3 + 2.\underline{6}.7 + 9.1.\underline{5} = 225$$

$$F_d = 4.\underline{8}.3 + 2.5.\underline{7} + 9.1.6 = 220$$

$$F_e = 4.7.3 + 2.5.8 + 9.1.6 = 218$$

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R. NORTH

*Editorial Note.* Solutions to this problem were also received from M. Ritter and J. C. Butcher.

## 2860. On note 2796

See also Note 2214 (May 1951).

The method of Mr. C. V. Gregg,—which can be described as the use of

$$(D_1 + D_2)^n = D_1^n + nD_1^{n-1}D_2 + \dots$$

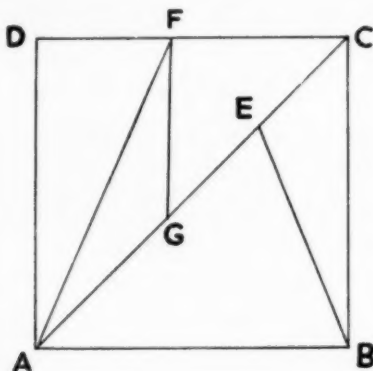
(as in Leibniz's theorem with  $n$  a positive integer) in cases where the index is  $-1$  or other negative integer to determine integrals of products,—seems to be moderately well known, although not noticed in the usual calculus text-books. The formula is given, with remainder, in Valiron, *Théorie des fonctions* (Paris 1948) p. 102, and used in the discussion of Taylor's theorem. It is convenient also in some integrals of the types arising in Fourier and Laplace transforms.

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C. WALMSLEY

**2861. A curious dissection of a square**

This dissection shows just one example of each of the different possible types of triangle whose angles are multiples of  $\pi/8$ . The pieces can be rearranged to form a kite and a few other shapes, but make no pretence to rival "Tangrams."



$$AB = BC = CD = DA = AE = 1; DF = EC = \sqrt{2} - 1;$$

$$FC = FG = AG = 2 - \sqrt{2}; AF = (4 - 2\sqrt{2})^{\frac{1}{2}}; EB = (2 - \sqrt{2})^{\frac{1}{2}}$$

C. DUDLEY LANGFORD

**2862. The distributive law for vector multiplication**

In an elementary treatment of vectors, there is a certain amount of difficulty in showing briefly and convincingly that

$$\mathbf{A} \wedge (\mathbf{B} + \mathbf{C}) = \mathbf{A} \wedge \mathbf{B} + \mathbf{A} \wedge \mathbf{C}. \quad (1)$$

A simple, short and completely elementary proof, requiring only a change in the usual order of procedure, is as follows. Immediately after defining the vector product, the scalar triple product  $(\mathbf{P} \wedge \mathbf{Q}) \cdot \mathbf{R}$  is introduced, and it is shown that

$$\mathbf{P} \cdot (\mathbf{Q} \wedge \mathbf{R}) = (\mathbf{P} \wedge \mathbf{Q}) \cdot \mathbf{R}. \quad (2)$$

This is done by the usual argument about the volume of a parallelepiped, together with consideration of the various special cases when the triple product is zero. Now, let

$$\mathbf{X} = \mathbf{A} \wedge (\mathbf{B} + \mathbf{C}) - \mathbf{A} \wedge \mathbf{B} - \mathbf{A} \wedge \mathbf{C}. \quad (3)$$

Then, making successive applications of the result (2), and the distributive law for scalar multiplication, which we suppose already

established, we have

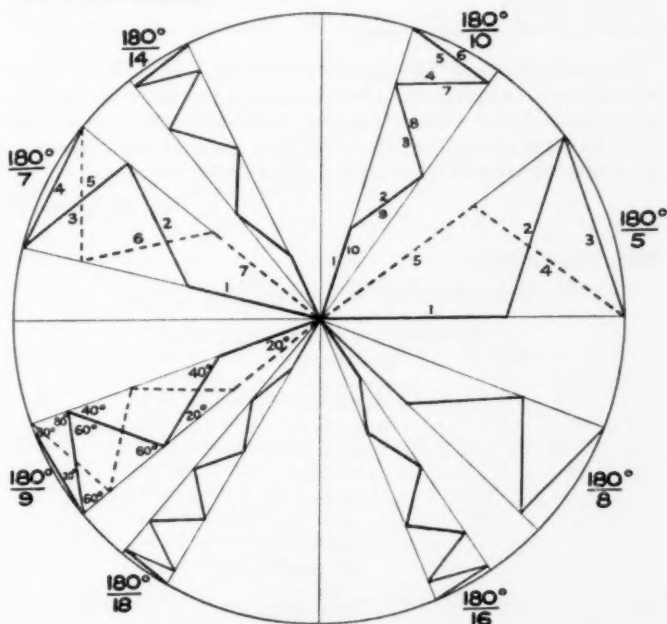
$$\begin{aligned}\mathbf{X} \cdot \mathbf{X} &= \mathbf{X} \cdot \{\mathbf{A} \wedge (\mathbf{B} + \mathbf{C})\} - \mathbf{X} \cdot (\mathbf{A} \wedge \mathbf{B}) - \mathbf{X} \cdot (\mathbf{A} \wedge \mathbf{C}) \\ &= (\mathbf{X} \wedge \mathbf{A}) \cdot (\mathbf{B} + \mathbf{C}) - \mathbf{X} \cdot (\mathbf{A} \wedge \mathbf{B}) - \mathbf{X} \cdot (\mathbf{A} \wedge \mathbf{C}) \\ &= (\mathbf{X} \wedge \mathbf{A}) \cdot \mathbf{B} + (\mathbf{X} \wedge \mathbf{A}) \cdot \mathbf{C} - \mathbf{X} \cdot (\mathbf{A} \wedge \mathbf{B}) - \mathbf{X} \cdot (\mathbf{A} \wedge \mathbf{C}) \\ &= 0, \text{ i.e., } \mathbf{X} = 0,\end{aligned}$$

and, from (3), the required result (1) is established. In addition to its brevity, this proof has the advantage of using only the methods of vector algebra, without recourse to the Euclidean geometry on which the usual text-book proofs depend.

*The University, Reading.*

D. H. PARSONS

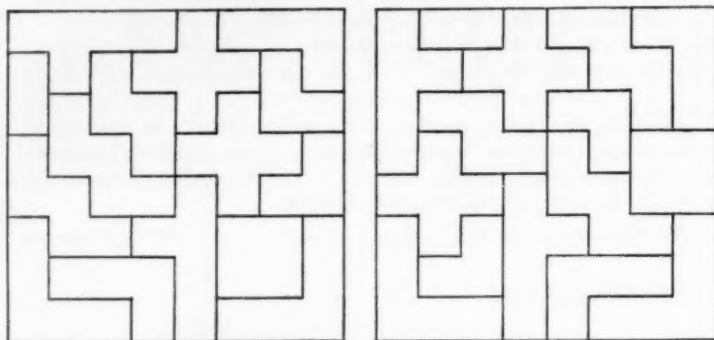
### 2863. Zig-Zag paths



The denominators of the above fractions represent the number of equal steps taken zig-zag between the sides of the various angles from the apex to the limit and back to the apex.

The outward and inward paths are different for the angles with odd denominators but are the same for the even ones.

ARCHIBALD H. FINLAY

**2864. A chess-board puzzle**

The pieces shown in the diagrams can be arranged to form a square with either side uppermost. If the squares of the underlying grid are coloured black and white alternately, with each white square on the back of a black square, then there is at least one more way of arranging them as a chess-board by turning some of the pieces over.

C. DUDLEY LANGFORD

**2865. Missing digits**

Find the missing digits in the following long division.

$$\begin{array}{r}
 \text{xxxxxx} \text{ x0xxxxxxxx00 (xxxxxx} \\
 \text{xxxxxx0} \\
 \hline
 \text{x0xx0xx} \\
 \text{xxxxxx0} \\
 \hline
 \text{x0xxxxx} \\
 \text{xxxxxx0} \\
 \hline
 \text{xx00xxx} \\
 \text{xxxxxx0} \\
 \hline
 \text{xxxxxx0} \\
 \text{xx0xx0x} \\
 \hline
 \text{xxxxxx0} \\
 \text{xxxxxx0} \\
 \hline
 \end{array}$$

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G. A. GUILLOTTE

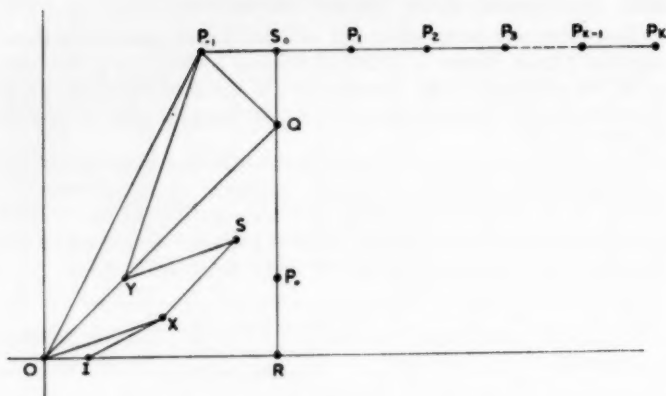
## 2866. Developments of the "Argand diagram"

Corrections are required in my article "Visual Aids in Modern Algebra" [*Math. Gazette* 41 (1957)]. Figures 2 and 8, on p. 247 and p. 242 respectively, were interchanged in the printing, each being given the other's caption; also, in the diagram on p. 242, 'N' should be read for 'K'.

This confusion may have obscured the fact that both figures are simple developments of the "Argand diagram." A synoptic display of such developments is given in the accompanying figure; several ways of adding and multiplying ordered pairs are illustrated in the diagram; the various algorithms are listed in the table below.

If $a, b, c, d$ represent	and $X(a, b), Y(c, d)$ represent	then their sum is represented by the point	and their product by the point
algebraic reals	complex numbers $a + bi, c + di$	$S(a + c, b + d)$	$P_{-1}(ac - bd,$ $ad + bc).$
elements of suitable rings $R_1, R_2$ , $a, c$ from $R_1$ , $b, d$ from $R_2$ , integers, $a \neq 0 \neq c$ ,	ring elements of the direct sum $R_1 + R_2$	$S(a + c, b + d)$	$P_0(ac, bd).$
	rational class members (fractions) $b/a, d/c$ ,	$S_0(ac, ad + bc)$	$P_0(ac, bd).$
naturals	integer class members,	$S(a + c, b + d)$	$P_1(ac + bd,$ $ad + bc).$
integers	integer differences $a - b, c - d$ ,	$S(a + c, b + d)$	$P_1(ac + bd,$ $ad + bc).$
rationals	polynomial residue classes mod $x^2 - k$ , where $k$ is a non- square positive integer, (surds) $a + b\sqrt{k}, c + d\sqrt{k}$ ,	$S(a + c, b + d)$	$P_k(ac + kbd,$ $ad + bc).$

$O$  is the origin,  $I$  the point  $(1, 0)$ ,  $X(a, b)$ ,  $Y(c, d)$ .  $XOYS$  is a parallelogram; triangles  $IOX$ ,  $YOP_{-1}$  are similar and similarly situated.  $P_{-1}Q$  is perpendicular to  $OY$  (produced if necessary). The vertical through  $Q$  meets the horizontal axis in  $R$  and the horizontal through  $P_{-1}$  in  $S_0$ .  $P_{-1}S_0 = S_0P_1 = P_1P_2 = P_2P_3 = \dots = P_{k-1}P_k$ . On  $RQ$   $P_0$  is taken so that  $P_{-1}S_0 = RP_0$ ,  $P_0$  being marked above or below the horizontal axis according as  $S_0$  is to the right or left of



$P_{-1}$  (that is, according as  $X$  and  $Y$  are on the same or opposite sides of the horizontal axis).

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T. DONNELLAN

#### 2867. Octahedron in a cube

The problem of inscribing an octahedron in a cube, proposed by Dorman Luke, *Math. Gazette*, Vol. XLI, p. 194 (1957) and solved by T. Bakos, *Math. Gazette*, Vol. XLIII, p. 17 (1959), appeared in the *Mathematics Magazine*, Vol. 25, Sept.-Oct. 1951, pp. 48-49.

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LEON BANKOFF

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1930. "It is a curious fact that pure mathematics should have been the most respected alternative to the so-called Humanities—the languages, history, literature and philosophy of two ancient civilizations—in the two ancient Universities as long as classical studies enjoyed first place; because, in truth, there is no more unhumane study than pure mathematics. No one doubts its value as a mental discipline, but its effect on its devotees is to incline them to a black-and-white view of men and affairs, where the answer is always right or wrong, and even when mathematicians apply their craft to science they overestimate the possibility of putting all variables in a problem into consistent and manageable mathematical terms. Perhaps the explanation of its favoured position is that pure mathematics is the main form of non-literary activity which can be carried on wholly in the polite seclusion of the study, and without any risk of dirty hands or of contact with less exalted mortals."—From 'Humane Studies,' by G. H. Rawcliffe. *Universities Review* 30 (1957) [Per Dr. B. H. Neumann.]

## CLASS ROOM NOTES

### 37. Limits

"If for each  $\varepsilon > 0$  there exists  $N$  such that for all  $n > N$   $|f(n) - a| < \varepsilon$ , then  $\lim_{n \rightarrow \infty} f(n) = a$ ."

The importance of the words "for all  $n > N$ " can be illustrated by considering the following two limits.

$$(i) \quad \lim_{n \rightarrow \infty} \sin^n \frac{n\pi}{99}$$

and

$$(ii) \quad \lim_{n \rightarrow \infty} \sin^n \frac{n\pi}{100}$$

W. W. O. SLESSENGER

### 38. The nine points circle

It is assumed that we have established the fact that the mid-points of the sides of a triangle and the feet of the altitudes lie on a circle.

The following proof shows that the mid-points of the lines joining the vertices to the orthocentre lie on the same circle.

In  $\triangle ABC$ ,  $D, E, F$  are the mid-points of sides,  $BC, CA, AB$  and  $P$  the foot of the altitude through  $A$ .  $V$  is the orthocentre, and  $X$  the mid-point of  $AV$ .

Draw the straight lines  $EF, ED$ , and produce  $EX$  to meet  $AB$  at  $J$ .

Since  $\triangle AFE$  is similar to  $\triangle ABC$ , and every length in  $\triangle AFE$  is half the corresponding length in  $\triangle ABC$ , therefore  $X$  is the orthocentre of  $\triangle AFE$  and so  $\angle FJE = 90^\circ$ . But  $ED$  is parallel to  $AB$ , so that  $\angle JED = 90^\circ$ . Hence  $\angle XED + \angle XPD = 180^\circ$  and  $X, E, D, P$  lie on a circle.

This proof was found by a Lower Sixth Form pupil, Miss F. Gee.  
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### 39. Cubic graphs

In Note 2635 Mr. A. Hurrell mentioned a useful property to know when sketching cubic graphs.

The following property is also interesting.

If a graph of a cubic function  $y = f(x)$  meets the  $x$ -axis in the points  $(a, 0), (b, 0), (c, 0)$ , then the tangent at the point  $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$  passes through  $(c, 0)$ .

Let

$$y = f(x) \equiv k(x-a)(x-b)(x-c),$$

then we have

$$f\left(\frac{a+b}{2}\right) = -k(a-b)^2(a+b-2c)/8,$$

$$f'(x) = k(3x^2 - 2(a+b+c)x + (bc+ca+ab)),$$

and therefore

$$f'\left(\frac{a+b}{2}\right) = -k(a-b)^2/4.$$

So that the equation of the tangent is

$$y + k(a-b)^2(a+b-2c)/8 = -k(a-b)^2\left(x - \frac{a+b}{2}\right)/4$$

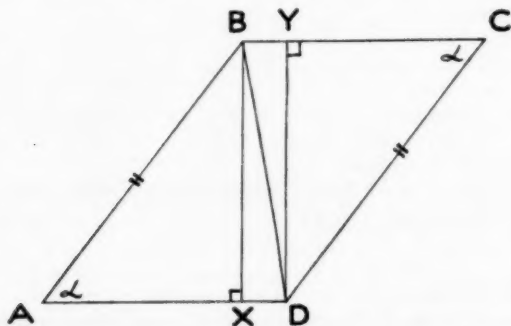
and  $x = c, y = 0$  satisfy the equation.

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T. NAKAZAWA

#### 40. The ambiguous case

In a quadrilateral  $ABCD$ ,  $\angle A = \angle C$  and  $AB = CD$ ; is the quadrilateral necessarily a parallelogram? Most people seem to think so, and I recently received the following "proof," which depends on how the figure is drawn.



Draw  $BX$  and  $DY$  perpendicular to  $AD$  and  $BC$ . Join  $BD$ .

$\triangle$ 's  $ABX, CYD$  are congruent

therefore  $BX = DY$  and  $AX = CY$ .

Hence  $\triangle$ 's  $BXD, DYB$  are congruent

and so

$$XD = YB.$$

Therefore

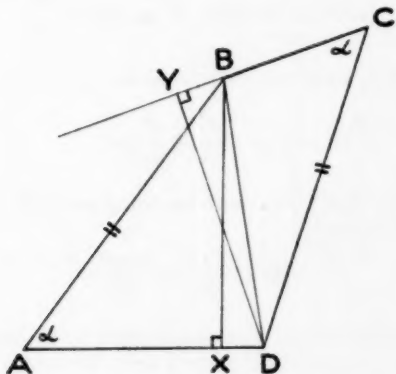
$$AX + XD = CY + YB$$

or

$$AD = BC$$

so that  $ABCD$  is a parallelogram ( $AB = CD$  and  $AD = BC$ )

This is correct if  $X$  and  $Y$  both lie within  $AD$  and  $BC$ , or if both lie in these sides produced ( $AX - XD = CY - YB$  in the latter case), but not if only one lies in a side produced, as in the following diagram.



$\triangle$ 's  $ABD$ ,  $CBD$  are the two triangles which can be drawn with two sides ( $AB$ ,  $BD$ , and  $CD$ ,  $DB$ ) and an angle not included ( $\angle A$  and  $\angle C$ ) equal to each other, and yet are not congruent (the ambiguous case).

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P. HALSEY

#### 41. Convergence on the Argand diagram

Two members of a first-year 6th form, rooting about in *Mathematics and the Imagination* for scraps of sensational news, found  $e^{i\pi} + 1 = 0$ , which (rightly) seemed to them to be a pearl of great price. Their poor fool of a tutor, whom they regard as a kind of performing animal, weakly (and with some misgivings) allowed himself to be red-herringed. The result was a period, previously assigned to the rather dull co-ordinate geometry of the circle, which developed on the following lines.

The complex number  $a + ib$  represents the vector from  $O(0, 0)$  to  $A(a, b)$ . The length  $OA$  is its modulus, the angle  $XOA$  its amplitude.

Let there be two vectors, one with modulus  $r$  and amplitude  $\theta$ , the other with modulus  $r'$  and amplitude  $\theta'$ . We may define their product as the vector with modulus  $rr'$  and amplitude  $\theta + \theta'$ —thus multiplication consists of stretch and turn.

Now  $i\pi$ , clearly an abbreviation for  $0 + i\pi$ , has modulus  $\pi$  and amplitude  $\frac{\pi}{2}$ . Consequently:—

$(i\pi)^2$  has modulus  $\pi^2$ , amplitude  $\pi$

$(i\pi)^3$  has modulus  $\pi^3$ , amplitude  $\frac{3\pi}{2}$

and so on.

We know that, when  $x$  is real, the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

converges to  $e^x$ . If  $e^{i\pi}$  is to mean anything at all, it must surely be the sum of the series

$$1 + (i\pi) + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \dots$$

This is a series of vectors. Let us add them by the usual process of coupling each to the end of the one before. The result is a polygonal spiral, tightening itself round the point  $(-1, 0)$ , as shown in Fig. 1.

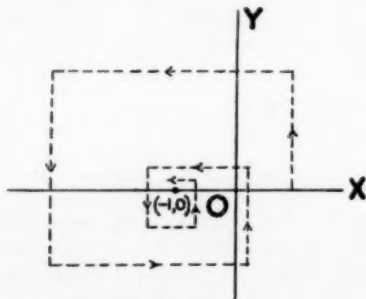


FIG. 1

The angles of the polygon are all right-angles; and its sides are, successively,  $1$ ,  $\pi$ ,  $\frac{\pi^2}{2}$ ,  $\frac{\pi^3}{6}$ ,  $\frac{\pi^4}{24}$ , etc. Some careful calculation and drawing are very much to the point.

This suggests a technique, which might be more widely used, for illustrating the convergence of series of complex terms. One can see the series converging, without recourse to the analysis into real and imaginary parts. The analysis, when it comes, will thus be illuminated and more readily grasped.

In the slightly more general case,  $e^{i\theta}$ , the spiral (Fig. 2) strangles the point  $A(\cos \theta, \sin \theta)$ . For  $e^{\sqrt{3}+i}$  we have Fig. 3, in which the

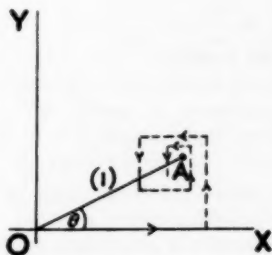


FIG. 2

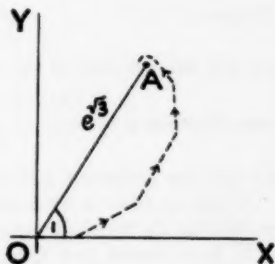


FIG. 3

exterior angles of the polygon are all  $\frac{\pi}{6}$ , while its sides are 1, 2, 2,  $\frac{4}{3}$ ,  $\frac{2}{3}$ ,  $\frac{4}{15}$ ,  $\frac{4}{45}$ , etc. In this case  $OA$  is about 5.65 and  $\angle XOA$  is 1 radian.

The geometric progression

$$1 + z + z^2 + z^3 + \dots$$

provides good examples. When  $z = -\frac{i}{2}$ , it gives a diminishing spiral, converging on  $\frac{4-2i}{5} \left( = \frac{1}{1-z} \right)$ . For  $z = 1+i$ , it gives an obviously divergent spiral. For  $z = \frac{1+i}{\sqrt{2}}$ , in which case the point (2) is on the circle of convergence, the figure is a regular octagon  $S_n$  having only 8 possible values and tending to no limit.

L. W. H. HULL

#### 42. The M-function: a note on inequalities of the means of a set of positive numbers

Several inequalities which appear in elementary algebra text-books are found to wear a disconnected air; and while it is not suggested that their algebraic treatment should be replaced by an analytical one, a coherence can be given to them by a second approach as follows:—

1. On the set of strictly positive numbers  $a_1, a_2, \dots, a_n$  not all equal, we define the real single-valued function  $y(p)$  of a real variable  $p$ , such that

$$(i) \text{ when } p \neq 0 \quad y^p = \frac{1}{n} \sum_{r=1}^n a_r^p$$

i.e.  $y$  is the  $p$ th root of the mean of the  $p$ th powers;

$$\text{and (ii) when } p = 0 \quad y^n = \prod_{r=1}^n a_r$$

i.e.  $y$  is the geometric mean.

When we wish to write the function in extended form in terms of  $p$  and the  $a$ 's we shall write it as  $M_p(a)$ , but otherwise as  $y = f(p)$ .

2. It is clearly seen from (i) that the function is continuous and differentiable for positive  $p$  and again for negative  $p$ . It will be sufficient for our purpose to show that it is *continuous* at  $p = 0$ , thus:—

Expanding (i) we have if  $p \neq 0$

$$1 + p \log_e y + O(p^2) = \frac{1}{n} \sum_{r=1}^n \{1 + p \log a_r + O(p^2)\}$$

$$\text{therefore} \quad \log_e y - \frac{1}{n} \sum \log_e a = O(p)$$

$$\text{and so} \quad f(p) \rightarrow f(0) \quad \text{as } p \rightarrow 0$$

It is immediately apparent that the geometric mean can be considered in relation to those of arithmetic type.

3. We shall now show by an analytical method that when  $p_2 > p_1$ ,  $f(p_2) > f(p_1)$ . [The more usual algebraic treatment is in Hardy, Littlewood and Polya's *Inequalities*, 2nd Ed. p. 28.]

$$\text{For } p \neq 0, \quad y^p = \frac{1}{n} \sum_{r=1}^n a_r^p$$

Differentiating w.r.t  $p$

$$y^p \log y + \frac{p}{y} \frac{dy}{dp} = \frac{1}{n} \sum (a_r^p \log a_r)$$

and therefore 
$$\frac{p^2 dy}{y dp} = \frac{1}{n} \sum a_r^p \log a_r^p - y^p \log y^p$$

$$= \frac{1}{n} \sum (b_r \log b_r) - \frac{1}{n} \sum (b_r) \log \left( \frac{1}{n} \sum b_r \right) \text{ where } b_r = a_r^p$$

$$= \frac{1}{n} \varphi(b_1, b_2, b_3, \dots, b_n), \text{ say.}$$

Consider first  $b_1 > b_2 > b_3, \dots, b_n$ , and let  $b_1 = b_2 + t$ .

Then

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial b_1} = \log \frac{nb_1}{S} > 0,$$

where  $S = \sum b_r$ , so that reducing  $t$  to zero reduces  $\varphi$ .

The process of telescoping the values of the  $b$ 's is carried out in  $(n-1)$  stages, finishing with  $\varphi = 0$ .

At the  $r$ th stage,  $b_1 = b_2 = \dots = b_r = b_{r+1} + t$ .

Then

$$\frac{d\varphi}{dt} = r \frac{\partial \varphi}{\partial b_r} = r \log \frac{nb_r}{S} > 0$$

where  $S$  is the new sum, and so at every stage  $\varphi$  is reduced.

If some but not all of the  $b$ 's are equal, the reduction process shows that  $\varphi > 0$ .

Thus  $\frac{dy}{dp} > 0$ , for  $p \geq 0$

and so if  $p_2 > p_1 > 0$  then  $f(p_2) > f(p_1) > f(0)$ .

and if  $0 > p_2 > p_1$  then  $f(0) > f(p_2) > f(p_1)$

so that finally for all  $p$  if  $p_2 > p_1$ , then  $f(p_2) > f(p_1)$ .

4. The various types of mean which are commonly considered now fall into line.  $M_3(a)$  is the cube-root of the mean cube,  $M_2(a)$  is the root-mean-square,  $M_1(a)$  is the arithmetic mean,  $M_0(a)$  the geometric mean, and  $M_{-1}(a)$  the harmonic mean. These are therefore in decreasing order of magnitude, a result which commends itself to the writer also in the form  $C > R > A > G > H$ .

[It should be noted that  $H$  is defined by  $\frac{1}{H} = \frac{1}{n} \sum_{r=1}^n \frac{1}{a_r}$ , in disagreement with the notation of Ferrar's Algebra (sequel).]

It is of interest to evaluate these means for a set of numbers. For the numbers 8, 9, 10, 11, 12 it is only just possible, working with 4 figure tables, to distinguish  $f(p)$  from a straight line between  $p = -1$  and  $p = 2$ . The values are therefore given for the set

3, 4, 5, 6, 7 whose relative spread is too large for the inequalities to show up finely. For this set

$$M_{-3} = 4.178; M_{-2} = 4.370; M_{-1} = 4.588;$$

$$M_0 = 4.789; M_1 = 5; M_2 = 5.196; M_3 = 5.371$$

These figures show that the function slows down for high and low values of  $p$ . The young student will probably see intuitively (a) that the function is intermediate in value between the greatest and least members of the set and (b) tends to these extreme values as  $p \rightarrow +\infty$  and  $-\infty$  respectively. The proofs of these are excellent exercises in algebra and analysis respectively.

5. It would be wrong to leave the discussion at this point, since inequalities are often required in the mean-power rather than the root-mean-power form; e.g. we have shown that

$$\sqrt{\frac{1}{n} \sum_{r=1}^n a_r^2} > \frac{1}{n} \sum_{r=1}^n a_r$$

and it follows that

$$\frac{1}{n} \sum_{r=1}^n a_r^3 > \left( \frac{1}{n} \sum_{r=1}^n a_r \right)^3$$

but if we are required to raise to a negative power the inequality is reversed, e.g. we have shown

$$\left[ \frac{1}{n} \sum_{r=1}^n \frac{1}{a_r} \right]^{-1} < \frac{1}{n} \sum_{r=1}^n a_r$$

Hence

$$\frac{1}{n} \sum_{r=1}^n \frac{1}{a_r} > \left[ \frac{1}{n} \sum_{r=1}^n a_r \right]^{-1}$$

and in general

$$\frac{1}{n} \sum_{r=1}^n a_r^p > \left[ \frac{1}{n} \sum_{r=1}^n a_r \right]^p$$

if  $p > 1$  or is negative, a result which presents some difficulty to the beginner, and which may perhaps be a little easier when approached in this fashion.

*St. Paul's School*

A. J. MOAKES

1931. *So much for the sand-reckoner.*—ARCHIMEDES asserted  $3\frac{1}{7} < \pi < 3\frac{1}{2}$ ; and his knowledge of geometry would have let him do better if only his arithmetic had been less immature.—Editorial note, p. 12, *Oxford Mathematical Conference, Abbreviated Proceedings*, 1957. [Per Mr. T. Donnellan.]

## REVIEWS

**An Emerging Programme of Secondary School Mathematics.** By M. BEBERMAN. Pp. 44. 7s. 6d. 1959. (Oxford University Press)

This little book describes the work which is being done by a group of teachers in the University of Illinois to develop material and methods of instruction to make mathematics more attractive and comprehensible to school children. Many mistakes in learning and teaching are traced to a confusion of number and numeral, and similar inaccuracies in the use of words; it is exceedingly interesting to find distinctions which have been the study of linguistic philosophy now proving to be important in elementary teaching. This is a book which every teacher of mathematics should read.

R. L. GOODSTEIN

**An Introduction to Coordinate Geometry.** By A. BARTON. Pp. 303. 21s. 1958. (University of London Press)

There are so many books on plane coordinate geometry that my first reaction was to doubt whether there could be any good reason for writing another. Having read the book I am converted. I cannot remember having read any other book in which the author takes so much trouble to anticipate the difficulties which a beginner may meet. To take one example, he explains that  $xy^2$  means  $x$  multiplied by the square of  $y$ , but  $XY^2$  means the square of the distance between the points  $X$  and  $Y$ . All through the book the author explains carefully what he is doing and (perhaps more important still) why he is doing it.

The early part of the book gives the impression that the author is hopping rapidly from one topic to another before finally settling down. However, this is deliberate: the intention is that the beginner should have an early introduction to some of the curves which he will study systematically later.

Most of the book is concerned with straight lines and conics (including the general equation of a conic). The last two chapters introduce some further curves. These are mostly curves which have simple geometrical definitions. These include familiar friends such as the Cissoid of Diocles and the Witch of Agnesi, but also some less familiar curves such as Watt's Lemniscate, which is defined by means of a simple linkage.

This is a book which can be strongly recommended to sixth-formers. It will be particularly valuable for those who have to do a good deal of the work by private study.

E. J. F. PRIMROSE

**Coordinate Geometry with Vectors and Tensors.** By E. A. MAXWELL. Pp. 194. 25s. 1958. (Clarendon Press: Oxford University Press)

Nowadays most people would agree that vectors should be used in three-dimensional geometry, but there are different opinions about the

point at which they should be introduced. Some believe that vectors should be used from the start. Others believe that it is best to develop the subject by the use of Cartesian coordinates, and to show later how vectors can be used to shorten the work. Dr. Maxwell takes the latter view, and I strongly agree with him. Beginners frequently cannot appreciate what the various vector quantities mean unless they already have some experience of the subject. Dr. Maxwell introduces vectors after the work on straight lines and planes. The later chapters are arranged so that either the Cartesian or the vector treatment can be followed.

The chapter on tensors is very well written. Even with the author's clear presentation to help them, beginners will not find the work easy, but later in their mathematical course they will find that it has been well worth doing.

There is one topic which might have been made easier for a beginner to grasp, namely the general quadric. The author starts by showing how to reduce a homogeneous quadratic form in three variables to a sum of squares by an orthogonal transformation, with no explanation of the geometrical motive. The result is that the algebra may seem rather artificial: in particular, why is the characteristic equation so important? If it had been previously explained that the aim was to find the principal diameters of the quadric, the characteristic equation would have appeared naturally.

First-year undergraduates will find the book most helpful. For many, it will provide a complete course, while those who need to go further will have been given a very clear introduction. Sixth-formers who are intending to study mathematics at a university would gain a great deal by reading at least the more elementary chapters.

E. J. F. PRIMROSE

**A First Course in Statistics.** By ROBERT LOVEDAY. Pp. xii + 128. 8s. 6d. 1958. (Cambridge University Press)

A First Course in Statistics is designed to meet the needs of boys and girls who are preparing to take statistics as an ordinary level subject for G.C.E. It is well produced, provides an interesting variety of examples and a useful selection of examination questions. Tables of logarithms, antilogarithms, squares and square roots are also given and the blank pages at the end of the book should come in handy for extra notes and formulae.

The explanations of method are generally clear; it is therefore unfortunate that the book contains a few statements which may be misleading to beginners. The explanation of skew given on page 6 seems to suggest that all symmetrical distributions are normal, although the note attached to Figure 4a might be taken as evidence to the contrary. The usefulness of the assumption that the average value of the items in a class is equal to the mid-value of the class interval depends on the shape of the distribution as well as the size of the classes. The method of calculating means which is described on page 26 is not suitable for highly skew or step distributions, although it is entirely adequate for

the type of problem quoted here. The explanation of the second regression line given on page 50 is unconvincing. If these lines are to be described as the loci of the means of the arrays, then theoretically there must be two regression lines. This point would be better understood if the first example of a bivariate distribution was given in the form of a two-way table and used data for which both regression lines were meaningful. Either of examples 11 or 12 on page 55 would have been a better choice than time series relating to numbers of vehicles licenced and casualties caused by road accidents.

The treatment of examples involving social statistics is less satisfactory. The description of a graph of annual numbers of births as the rising birth-rate seems to invite confusion on this subject. So does the explanation of the age structure of the population on page 13. Movable holidays such as Easter and Whit complicate the seasonal pattern of the Index of Industrial Production: the seasonal variation in this index cannot be eliminated by a simple application of the method of moving averages. The weighting systems used for retail price indices usually describe the pattern of expenditure for large groups of persons at particular times, but the weights for the 1947 index were derived from an analysis of household expenditure for 1937-38 and information concerning prices in 1947. Table 9E should be interpreted with great caution. The years 1947-56 were years of rising prices; personal expenditure on most goods and services, including drink and tobacco, was higher at the end of the period than at the beginning, though the proportion of total expenditure allocated to individual commodities changed during these years.

In his preface, Mr. Loveday suggests that his book might have been described as Stage A Statistics, since "like Stage A Geometry it is numerical, experimental and practical." Certainly this course requires little algebra or geometry; it suggests a few experiments with dice and the collection of some school statistics; but it is doubtful if the book as a whole merits the adjective practical. A price index obtained by combining price relatives for 1926 ( $1925 = 100$ ) with weights derived from an enquiry made in 1953-54 can only be meaningless. Also why should anyone want to calculate a 12-month moving average for data given for so short a time as 20 months, except of course, to convince the examiner that he knows how to do it!

FREDA CONWAY

**Certificate Mathematics.** By C. V. DURELL, M.A. Vol. I. pp. xvi + 320. 8s. 6d. With Answers (pp. xl). 9s. 3d. Vol. II. pp. xl + 320. 8s. 9d. With Answers (pp. xxxii). 9s. 6d. Vol. IIIA. pp. xl + 312. 9s. With Answers (pp. xxxii) 9s. 9d. Vol. IIIB. pp. xl + 288. 9s. With Answers (pp. xxxii) 9s. 9d. Vol. IVA. pp. xlii + 310. 9s. With Answers (pp. xxxii). 9s. 9d. Vol. IVB. pp. xlii + 294. 9s. With Answers (pp. xxxii). 9s. 9d. (Bell)

The author claims our respect as one of the most prolific and well established writers of mathematical texts for schools. His long experience at Winchester has no doubt influenced him, in preparing this

course, to concentrate more on the brighter pupil and in particular on the pupil who will continue with mathematics after 'O' level. Already at the beginning of Volume II in the list of formulas appear those for the area and the tangents of the half angles of a triangle in terms of the sides, as well as the sine, cosine and tangent formulas. Perhaps a good example of his appeal to the brighter pupil is the chapter on Pythagoras in Vol. IIIA where the Pythagorean figure is discussed both for a right angled and an acute angled triangle and from the latter is deduced the cosine formula. This gives added insight to the bright but confuses the weak. A discriminating teacher, however, should be able to use this book to good effect.

"The primary object of a homogeneous course," the author says, "is to encourage the pupil to select on every occasion the most appropriate method, whether algebraical or geometrical or graphical or trigonometrical." Although different methods are given, often in chapters far removed from one another, no attempt is made to discuss their relative advantages, and many a pupil could be left with a confusion of ideas and a mass of undigested detail.

Children use examples and exercises rather than the text, whereas the work of the teacher is to generate in their minds the relative mathematical ideas in order for them to think from as many points of view as possible in tackling their own work. From this angle the great number of problems, whether in the form of exercises, quickies, computation tests or revision papers, provides the teacher with a vast amount of material from which to choose, and enables him to grade the work to the abilities of different sets and to revise the work covered in previous years.

In particular the approach to geometrical study, both plane and solid, and the graphical introduction to the calculus are impressive for their simplicity, homogeneity and completeness. The arithmetic of the later volumes is, of necessity, disjointed, which stresses the fact that so much of our school arithmetic is concerned with application to various commercial topics rather than progress in ideas. A similar danger exists to a lesser extent in the algebraic work where the divorce of technique, practiced for its own sake, from application leads to emphasis on method rather than on purpose and robs introductory work of its simplicity and reality.

L. G. HURDIDGE

**Classbook of Arithmetic and Trigonometry.** By S. F. TRUSTRAM and H. WHITTLESTONE. Pp. 368. 12s. 6d. 1958. (G. Bell & Sons)

This book is a companion volume to Trustram's "A Classbook of Algebra," and consists almost entirely of exercises. It is in two parts which are issued separately or as one volume.

The early chapters are devoted to many graded exercises on the Four Rules for practice purposes and the diagnosis of weaknesses. Very little bookwork is given, the questions are realistic and up to date, and generally speaking the odd numbered examples are easier than the even, and each exercise contains one or two examples that can be

used for discussion on the blackboard. Plenty of examples are given on each point so that practice can be obtained, but it is not the author's intention that all the examples should necessarily be taken en bloc, but used with discretion in short periods over several terms work.

Much of the arithmetic required for 'O' level is covered in Part I as well as an introduction to Trigonometry. Part II comprises mainly work on Logarithms, more difficult Mensuration (including similar figures), Investments and the remainder of the Trigonometry required for 'O' level. There are 40 useful revision papers, and a glossary; the usual four figure Tables are supplied at the beginning of the book and Answers at the end. It should prove a most useful source of examples for many teachers.

B. J. F. DORRINGTON

**Oxford Graded Arithmetic Practice: Book 8 Fractions.** By D. A. HOLLAND. Pupils' Book 2s. 6d. Teacher's Book 3s. 6d. each 64 pages. 1958. Oxford University Press

The earlier books in this series were reviewed in *Mathematical Gazette*, Vols. XL and XLI. This book deals with common and decimal fractions (including percentages). The usual careful grading of mechanical examples is shown here and teachers who are looking for more exercises than are usually given in text books will most probably find what they need here. The book is recommended for upper junior and lower secondary modern forms. It does not treat multiplication and division of decimals by decimals nor examples using percentages of any greater difficulty than 72 min. is  $\frac{1}{6}$  of  $\frac{1}{2}$  hrs.; fractions however are dealt with fully including examples of the type  $10\frac{3}{4} \div 12 \times \frac{5}{8}$ .

K. SOWDEN

**Advanced Mathematics for Technical Students. Part II.** By LOWRY and HAYDEN. Pp. 423. 21s. 1957. (Longmans Green & Co.)

This is the second edition of a book which was first published in 1950. It is the second half of a two-volume work which covers, with the exception of statistics, theory of errors and the method of least squares, the requirements of the syllabus of Mathematics for the first two parts of the London B.Sc. (Eng.) Degree and should also prove useful as a text book for the Mathematics which is ancillary to Degrees in Physics and Chemistry, although the illustrative examples are nearly all related to Engineering problems.

The students for whom the book is written are primarily interested in the use of Mathematics for the solution of their problems and not necessarily in Mathematics itself; judged from this point of view the authors have produced a sound and extremely useful book. The treatment is sufficiently rigorous to make a student realize that there is more in Mathematics than the mere ability to apply a rule; on the other hand, the rigour is not carried to such lengths as would be appropriate in an Honours school of Mathematics.

In writing this second edition it seems a pity that the opportunity was

not taken of making a slight rearrangement in the section dealing with matrices. On page 157 the student reads "There is no question of the numerical value of a matrix," on page 160 he meets the matrix equation  $Z = 0$ , but the meaning of this statement is not explained until page 161 is reached.

A welcome feature of the new edition is the replacement of the operational methods of Heaviside by those of the Laplace transform. This improvement might be carried a stage further in a future edition by supplementing the useful information on vectors by illustrative examples of their use in solving problems in mechanics and three-dimensional co-ordinate Geometry.

F. T. CHAFFER

**Concise Physics.** By R. G. SHACKEL. Pp. 663. 17s. 6d. Second edition, 1958. (Longmans & Green)

This 'O' level book first published in 1938 has some additional technical material but some defects in general qualitative discussions remain (e.g. in viscosity, which need not be included) and the photometry units are obsolete. Real-is-positive sign convention is used.

The book has a workmanlike treatment of the basic topics, with good diagrams and questions: its faults would be extremely easy to remedy in the classroom.

A. J. MOAKES

**Introduction to Difference Equations.** By S. GOLDBERG. Pp. 260. 54s. 1958. (Chapman & Hall)

This introduction to the theory and application of difference equations is designed for readers with little more than a fifth form knowledge of mathematics and leads by easy stages up to a study of  $2 \times 2$  matrices. It is beautifully written, and gives insight not only into mathematical techniques but also into fields of application chosen from Economics and the Social Sciences. A good book for the mathematical specialist to find in the school library in his last year of school.

R. L. GOODSTEIN

**Tables Trigonométriques.** By C. FAUCHER. Pp. 54. 500 fr. 1957. (Gauthier-Villars, Paris)

The full title of this work is *Tables Trigonométriques contenant les valeurs naturelles des sinus et des cosinus de centigrade en centigrade du quadrant avec dix décimales*. Like the very brief preface, it appears also in English. The author is an experienced topographical surveyor who has already produced a notable table of logarithms. In the present work he provides sines and cosines on the centesimal system at interval one hundredth of a grade (a grade being one hundredth of a right angle). The values are to ten decimals, or rather slightly more, because there is indication of rounding (an extra binary digit, so to speak; that is, about a third of a decimal place). First differences are given, but not

second differences, which reach a maximum of 247 final units. Repeated digits are not always given, so that the differences have an uneven appearance, and care is needed in extracting them. The tables are adequately reproduced from manuscript. The computations were performed to twelve decimals without machine, by interpolation in tables of Andoyer, using multiplication tables of Crelle and of Peters.

This is a contribution of some importance because, surprising as it may seem, no more precise values appear to have been published at interval of one hundredth of a grade. Ten-decimal sines and cosines (also tangents, up to 50 grades) were published, along with first differences and indication of second differences, in a volume entitled *Sintacos 10* by V. Elznic at Prague in 1941, but a publication of Gauthier-Villars must in the post-war world be considered to possess some advantage in respect of accessibility to English-speaking mathematicians.

Unpublished values at the same interval were computed to 22 decimals under the supervision of Prony at the epoch of the French Revolution, and to 15 decimals by Sang, who died in 1890. It is to be hoped that someone will compare Faucher's tables with Sang's, which are contained in two of the many manuscript volumes due to this masterly computer which are preserved in the archives of the Royal Society of Edinburgh.

A. FLETCHER

**Spheroidal Wave Functions.** By C. FLAMMER. Pp. ix, 220. 68s. 1957. (Stanford University Press, Stanford, California)

This volume, which inaugurates a series of monographs of the Stanford Research Institute, aims at giving a unified account of sufficient theory for practical applications, together with a set of useful tables. Some unification is clearly desirable, for the subject of spheroidal wave functions stands to gain from some tidying up; the formulae and expansions tend in any case to be rather complicated, the literature is by now fairly extensive, and there is a minimum of uniformity about such things as notation and method of normalization. The comparative tables on pages 14 and 15 will help in sorting out the symbolisms and functions chiefly used. About two-fifths of the volume is given to theory and a list of 103 references, while the tables occupy pages 88 to 220. The tables have been compiled partly from a number of existing works (by Stratton, Morse, Chu and Hutner; Meixner; Spence; Leitner and Spence; and the Institute of Numerical Analysis at Los Angeles), with some recomputation, correction and adaptation where necessary, and partly from original work done at the Stanford Research Institute. Some of the computations were performed in connection with the author's own researches on physical problems. One should not be misled by the author's modest description of his tables as a hodgepodge; it seems to the reviewer to be distinctly useful to have collected tables, not uncritically, from sources of diverse degrees of accessibility (at any rate in this country), supplemented them with original tables, and arranged the whole in a clear and comprehensible fashion.

While the volume under review was in preparation, there appeared another, also entitled *Spheroidal Wave Functions*, by Stratton, Morse,

Chu, Little and Corbató. This monumental work, published in 1956, was recently reviewed in the *Gazette*. Broadly speaking, it abandons (whereas Flammer largely retains) the notation and normalization due to Chu and Stratton, and moves closer to the system of Morse and Feshbach. The lack of coincidence in notation and normalization is unfortunate. However, Flammer's text is in admirable notational correspondence with his set of tables. Compared with the recent tables of Stratton, Morse and their coworkers, the tables of Flammer give a less dense tabulation over a wider field (for example, Flammer includes values of functions, including those of the second kind, as well as values of coefficients in expansions). In spite of some misprints, Flammer's book appears to the reviewer to contain the best integrated set of formulae and tables available. It may certainly be recommended as a mine of information to any applied mathematician who requires a general view of the whole field, together with tables which may be expected to be of considerable use in dealing with a variety of physical and engineering problems. The tables for the prolate case are considerably fuller than those for the oblate one, in accordance with the inequality in the materials available to the author.

A. FLETCHER

**Collected works of Bernhard Riemann.** Edited by H. WEBER. Pp. 558 + 116. 23s. 1957. (Dover, New York. Mayflower, London).

The German language text is a reprint of the Second Edition (1892) of Riemann's Collected Works together with a 100 page supplement, and a new introduction (in English) by Hans Lewy. The Introduction gives a brief account of Riemann's life and an analysis of his mathematical discoveries "which have exerted a profound influence on the development of mathematics up to our day".

R. L. G.

**Set Theory.** By F. HAUSDORFF. Translated by J. R. Aumann, *et al.* Pp. 352. 1957. (Chelsea, New York).

This very welcome translation of Hausdorff's *Mengenlehre* has been made from the third edition (1937) of this justly famous book, which for more than forty years has served to introduce students to the arithmetic of aleph and omega, the theory of linear spaces, the fundamental concepts of point sets, compactness and connectedness, topological mappings and functions. Those who are familiar with the first edition (1914), which was reprinted by Chelsea in 1955, will regret, albeit unfairly, that the translation does not include some of the material which is contained only in the first edition, for instance the Jordan Curve Theorem and the specification of a topological space by neighbourhood axioms.

The omission from the second and later editions of any discussion of questions on the foundations of set theory, and of the paradoxes, has the obvious danger that the uninitiated student is left to find out for himself that the naive theory he is studying is inconsistent; he is not

even warned that the theorem of well ordering which plays so important a part is based on a concealed axiom. There are references in the Bibliography to recent work in the foundations of set theory but an editorial note would have been a more valuable addition.

The translation is the work of several hands, but it reads smoothly and the warm pedagogic style of the author has been faithfully preserved.

R. L. GOODSTEIN

**Zagadnienia Rozstrzygalnosci.** By A. GRZEGOREZYK. Pp. 141. 1957. (Warsaw).

This introductory survey of mathematical logic treats recursive functions, both primitive and general, the enumeration of primitive recursive functions, recursive arithmetic and the concept of an algorithm.

R. L. G.

**La Crise de la Raison et La Logique.** By E. W. BETH. Pp. 50. 900 fr. 1957. (Gauthier-Villars, Paris).

In addition to brief observations on the relation of mathematical logic to philosophy this book presents an interesting new method of inference, called the method of semantic tables which has affinities both to the Gentzen sequence calculus and to the use of semantic models. The method is applied to prove in detail a theorem of elementary geometry and it is shown how this proof may be transformed into a formal derivation. A formal system  $F$  is constructed such that a sequence

$$v_1, v_2, \dots, v_m \vdash V_1, V_2, \dots, V_m$$

is derivable in  $F$  if and only if its semantic table is closed. System  $F$  is proved to be complete and to admit the Gentzen 'Hauptsatz'.

R. L. G.

**1932.** It was a most interesting, valuable and educational experience to see Cayley solve a problem. He did not seem to trouble much about choosing the best method, but took the first that came to his mind. This led to analytical expressions which seemed hopelessly complicated and uncouth. Cayley, however, never seemed disconcerted but went steadily on, and in a few lines had changed the shapeless mass of symbols into beautifully symmetrical expressions, and the problem was solved. As a lesson in teaching one not to be afraid of a crowd of symbols, it was most valuable.—J. J. Thomson, *Recollections and Reflections* (1936), p. 47. [Per G. N. Watson; suggested by Note 2641.]

**Middel-algebra.** By P. WIJDENES. Pp. 419. f. 19. 1957. (Noordhoff. Groningen).

This school text covers very much the same ground as the larger English Algebra texts. The book starts with an account of induction and proceeds to consider inequalities, an arrangement which so to speak deals with hard things first. Then follows the theory of permutations and combinations, and the binomial and multinomial theorem. The fourth Chapter deals with the summation of series, and the next two present a comprehensive study of determinants and linear equations. Complex numbers are introduced as ordered pairs of real numbers and there is a detailed account of cubic and quartic equations. Since the aim of the book is to prepare the reader for a study of analysis there is a long section on the function concept. The derivative is introduced by a limiting operation and used to discriminate between maxima and minima and in the generation of Sturm's functions. The book concludes with chapters on root approximation, symmetric functions, the theory of elimination and partial fractions.

R. L. GOODSTEIN

**Applied Probability.** Vol. VII. Proceedings of the Seventh Symposium in Applied Mathematics of American Mathematical Society. Edited by L. A. MACCOLL. Pp. v, 103. \$5.00. 1958. (McGraw-Hill)

At the Seventh Symposium in Applied Mathematics held in April 1955 and concerned with *Mathematical Probability and its Applications* there were purposively three main themes—the theory of diffusion, the theory of turbulence, and probability in classical and modern physics. The Editor remarks that these themes could have been interpreted broadly if the participants of the symposium had so wished. Actually they seem to have kept approximately to the chosen topics. There are nine papers making up the volume, the papers being presented in the order in which the nine participants spoke. The first paper by Paul Lévy is entitled "Brownian Motion depending on  $n$  parameters; the particular case  $n = 5$ ". Most of the results in this paper have been stated without proofs in the *Comptes Rendus*. In this particular paper Professor Lévy gives the proofs and opens up a new chapter in probability theory by consideration of a random function which has two distinct expressions connected with two distinct stochastic differential equations.

In the second paper J. L. Doob has "A New Look at the First Boundary-Value Problem." The mathematical development here is kept to a minimum since he is not concerned with proofs but with ideas. This paper will probably be of value when allied to the previous work of Professor Doob and with those papers which this contribution will generate. From W. Feller we have "On Boundaries defined by Stochastic Matrices," a short paper (abstract) connected with the mathematical development of the random walk. Papers 4 and 6 by E. Hopf on "The Theory of Turbulence" and by G. K. Batchelor on "The Singularity in the Spectrum of Homogeneous Turbulence" use probability terminology and concepts but are not really probabilistic although they do seem to be real contributions to the theory of turbulence.

The paper by G. Münch (5) on "Stochastic Processes in Astronomy" is possibly that out of the whole volume which will appeal to probabilists working in the field of statistics in view of the recent discussion at the Royal Statistical Society on the statistical approach to problems of cosmology. It would be interesting to see the numerical evidence on which the author has based the mathematical concepts he requires to set up the stochastic process. This is perhaps done elsewhere for since this is a mathematical symposium only mathematical models and derivations therefrom are given.

The remaining three papers—M. Kac "Probability in Classical Physics," S. M. Ulan "Infinite Models in Physics" and B. O. Koopman "Quantum Theory and the Foundations of Probability" are on the third of the three themes set for the participants. Professor Koopman is thought-provoking in his reformulation of probability if not entirely convincing.

The brief mention made by the reviewer of the different papers which compose this symposium volume is probably enough to indicate that not all probabilists will be interested in all the papers although all will find something to interest them. Statisticians will not be interested in the contributions with the exception perhaps of paper 5.

F. N. DAVID

**Neutron Transport Theory.** By B. DAVISON and J. B. SYKES. 75s. 1957. (Oxford University Press)

Neutron Transport Theory is associated mainly with the nuclear energy world, including both nuclear reactors and nuclear weapons; it is also of theoretical importance in astro-physics. The theory was developed mostly during the last war under the great pressure of the atomic bomb project and the subject was an intensely practical one, with numerical results wanted quickly; the same applies now, so the development is and always has been closely linked with computing facilities.

The book gives a very detailed mathematical treatment of every known method of any importance (and some no longer of much importance) for the solution of neutron transport problems. It is undoubtedly unique, something of a tour de force and will certainly become a standard reference work. Davison's achievement in assembling all this material and putting it between one pair of covers is very impressive. I personally have three general criticisms. First, it is difficult to see the wood for the trees; although the book is essentially one of methods, it does not make clear what are the fields of application and the relative advantages and disadvantages of the numerous methods described. I feel that a general descriptive chapter on these lines would have been a valuable addition. Second, I think more should have been said about the computational aspects and the impact of the digital computer, for after all, users of neutron transport theory are always looking for numerical solutions and electronic computers are now in general use in this field. For example, one will find no indication in the book that multi-group calculations based on diffusion theory, quite impossible without powerful

computing machinery, are being done on a production basis in many establishments using direct numerical methods based on finite-difference approximations to the partial differential equations; this kind of approach, of course, side-tracks all the algebraic complications of the analytical methods. Nor does it seem to me the power of the Monte Carlo method of solution is brought out sufficiently strongly in the one chapter on this process. Finally, whilst one cannot expect a book of this type to be easy reading I think some more concessions could have been made to the reader; the verbal arguments needed to establish a point are often quite complicated and are not made easier by the need to chase back through reference after reference.

J. HOWLETT

**The Taylor Series.** By P. Dienes. Pp. 552. 22s. 1957. (Dover, New York; Constable, London)

This important book has been out of print for many years and the Dover reprint is very good value at 22s. The chapter on divergent series has inspired a good deal of research in the past twenty-five years. Another valuable feature of the book is a very full treatment of topological questions, and the singularities of analytic functions.

R. L. G.

**Tables of the Non-central  $t$ -Distribution.** By GEORGE J. RESNIKOFF and GERALD J. LIEBERMAN. Pp. 389. 100s. 1957. (Stanford University Press. London: Oxford University Press)

If  $Z$  is a normally distributed random variable with zero mean and unit standard deviation and  $W$  is an independent random variable such that  $fW$  has a  $\chi^2$  distribution with  $f$  degrees of freedom, the probability density function of  $T = (Z + \delta)/\sqrt{W}$  ( $\delta$  being some constant) is given by:

$$h(f, \delta, t) = \frac{\Gamma(f)}{2^{1/2} \Gamma(\frac{f}{2}) \sqrt{\pi f}} e^{-\frac{f\delta^2}{2(f+t^2)}} \left(\frac{f}{f+t^2}\right)^{\frac{f+1}{2}} H_h\left(\frac{-\delta t}{\sqrt{f+t^2}}\right)$$

where  $H_h(y) = \int_0^\infty \frac{v^f}{\Gamma(f)} e^{-(v+y)^2/2} dv$ , a function which was discussed and tabulated by Airey in 1931.

In this volume the following functions are tabulated:

- (i)  $h(f, \delta(p), x\sqrt{f})$  to 4 d.p.
- (ii)  $\int_{-\infty}^{x\sqrt{f}} h(f, \delta(p), t) dt = H(f, \delta(p), x\sqrt{f})$ , to 4 d.p.
- (iii) The solution,  $x(t, \delta(p), \epsilon)$ , of the equation

$$\int_{x\sqrt{f}}^\infty h(f, \delta(p), t) dt = \epsilon \text{ to 3 d.p.}$$

Here  $\delta(p)$  is the solution of the equation

$$\int_{\delta(p)/\sqrt{f+1}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = p.$$

All these functions are tabulated for  $f = 2(1)24(5)49$  and  $p = 0.0010, 0.0025, 0.0040, 0.0100, 0.0250, 0.0400, 0.0650, 0.1000, 0.1500$  and  $0.2500$ , and an auxiliary table of  $\delta(p)$  to 6 d.p. for these values of  $f$  and  $p$  is given. The range of  $x$  in tables (i) and (ii) varies for different values of  $f$  and  $p$ , but values of the functions are given at intervals of 0.05 in  $x$  over a range sufficiently large to include all values for which  $h \geq 0.00005$  and  $0.00005 \leq H \leq 0.99995$ . In table (iii) the function  $x$  is tabulated for  $\epsilon = 0.005, 0.01, 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95, 0.99$  and  $0.995$  for  $f \geq 5$ . For  $f = 2, 3$  and  $4$  the two smallest values of  $\epsilon$  are omitted.

An introduction illustrating the use of these tables is also included.

The publishers state that this volume of tables "affords the analyst the opportunity to make numerical investigations about functions of the non-central  $t$ -statistic. These investigations previously required the use of empirical sampling techniques." This is not strictly accurate. While the existing tables of the  $Hh$ -functions are in many cases not suitable for such investigations, it was often possible to use tables of the confluent hypergeometric function, in terms of which the  $Hh$ -function may be expressed. It would, however, be true to say that these tables will make investigations of this kind a great deal easier.

F. DOWNTON

**Selected Papers in Statistics and Probability.** By Abraham Wald. Edited for the Institute of Mathematical Statistics. Pp. x, 702. 80s. Second Impression, 1957. (Stanford University Press. London: Oxford University Press)

The tragic death of Abraham Wald in an aeroplane accident in 1950 ended the career of a man, whose contributions to the mathematical theory of statistics were many and varied and also of unquestioned quality.

Wald's interest in statistics began only in 1938, when he left Vienna and went to the United States where he obtained a fellowship at Columbia University. His previous interests had been in pure mathematics (largely geometry) and econometrics, although a few papers in probability theory had prepared the way for his work in the statistical field. That he quickly became at home there is shown by the fact that he almost immediately published papers on statistical theory. In fact, the paper which laid the foundations for one of his two best known contributions was published in 1939. This paper, entitled "Contributions to the theory of statistical estimation and testing hypotheses" was, in its essentials, the basis for his book, *Statistical Decision Functions*, which was not to appear until 1950, just before his death. The ideas behind this work are admittedly not universally accepted, but no professional statistician can afford to be ignorant of them.

His other substantial contribution was in the development of sequential methods of taking observations. Many of the papers he published

on this subject were included in his book, *Sequential Analysis* (published in 1947), and have been omitted from the present collection. It may be noted that sequential methods, based on Wald's use of the likelihood ratio, are already widely used by experimenters. In contrast, decision functions are a rarity in a practical statistician's armoury, although they were undoubtedly a more fundamental contribution to statistical theory.

In addition to these two main streams of Wald's statistical work, this volume contains other papers on such varied topics as non-parametric inference, the analysis of variance and the solution of stochastic difference equations. Finally, it contains a previously unpublished paper (except, that is, for the first impression of the present book, published in 1955) entitled, "Testing the difference between the means of two normal populations with unknown standard deviations." This is a third solution to the Fisher-Behrens-Welch problem.

Although sometimes hard to read, Wald's papers invariably contain, within their strict mathematical rigour, some stimulating statistical idea, and it is regrettable that this well-produced memorial to him was necessary so prematurely.

F. DOWNTON

**Proceedings of the First International Conference on Operational Research.** Pp. 526. 50s. 1958. (English Universities Press)

This volume of over 500 pages contains 28 papers on the function, the methodology, and the applications of Operational Research. There are also accounts on O.R. in 16 different countries (21 were represented at the conference), shortened versions of the discussion after the papers and of informal panel meetings.

The papers are arranged in the order in which they were read on 4 days. Mathematicians might be more interested in those of the second day, on such topics as gaming, linear programming, queueing, and inventory control.

The Editorial Committee deserves high praise for having produced this volume so quickly after the successful and splendidly organized conference. Of course, as in most such ventures, the original refereeing appears to have been rather sketchy.

This collection of papers, some from the pens of the most distinguished operational research workers, is a welcome addition to the libraries of many individuals and institutions.

S. VAJDA

**Vector Spaces and Matrices.** By R. M. THRALL and M. TORNHEIM. Pp. 318. 54s. 1957. (Chapman and Hall)

This is an excellent book, written in the modern spirit throughout, and can be warmly recommended for University Honours courses. It provides just the material that a course in Linear Algebra should contain. The theory of vector spaces comes first and is developed over an arbitrary field; the study of linear transformations precedes that of

matrices, determinants are introduced after the treatment of linear equations and are used most sparingly. The discussion always starts from the concrete level of the reader's presupposed limited mathematical experience and leads up to the axiomatic level. The requisite amount of abstract algebra in the theory of fields, rings, and groups is developed on the way. After the introductory chapters the authors study the usual equivalence relations and canonical forms such as similarity, congruence and scalar products, orthogonal and unitary equivalence etc. A final chapter deals with linear inequalities which are of importance in the theory of programming. Many examples and exercises, carefully chosen, add to the value of the book.

K. HIRSCH

**Theory of Functions of a Real Variable.** By I. P. NATANSON. Translated by L. F. BORON. Pp. 277. 40s. 1958. (Ongar Co., New York. Constable & Co. London)

This translation has been made from the first nine chapters of the original Russian Edition with additional notes by E. Hewitt. The German translation was warmly welcomed by Prof. Rogosniski in his review (*Gazette* XL, p. 146) and some personal observations in the original text to which he took exception have been omitted. Third year students reading Analysis in British Universities will find this book extremely helpful; the author has a warm and genial style and lengthy proofs are broken up into parts to make for easy reading. The book presupposes a knowledge of, and some familiarity with the Riemann integral but builds up the requisite set theory (on an intuitive basis) *ab initio*.

R. L. G.

**Arithmetik.** By P. B. FISCHER. 3rd Ed. Pp. 152. DM 2.40. 1958. (W. de Gruyter & Co., Berlin)

A brief account of operations on natural numbers, integers, real and complex numbers.

R. L. G.

**The Scientific Papers of Sir Geoffrey Ingram Taylor.** Vol. I. Mechanics Solids, Edited by G. K. BATCHELOR. 75s. 1958. (Cambridge University Press)

This is the first of a series of four volumes containing the collected works of Sir Geoffrey Taylor who has been a pioneer of research in the mechanics of solids and fluids. Volume I contains all Taylor's studies on the mechanics of solids and includes papers on elasticity, plasticity, the properties of metals, and dislocation theory. These collected works are indispensable to all workers in continuum mechanics.

A. E. GREEN

**Fourier Analysis and Generalized Functions.** By M. J. Lighthill. Pp. 79. 17s. 6d. 1958. (Cambridge University Press)

In the 20 years since Laurent Schwartz created the theory of distributions the simplifications introduced by Mikusinski and Temple have made possible an account of generalized functions at the undergraduate level. Lighthill's text is remarkable for its clarity and simplicity and may be warmly recommended to all students of mathematical physics.

R. L. G.

**The Numerical Solution of Two-point Boundary Problems in Ordinary Differential Equations.** By L. Fox. Pp. xi, 371. 60s. 1957. (Oxford University Press)

Numerical solutions of ordinary differential equations fall into two broad classes, the initial-value type and the boundary-value type, depending respectively upon whether boundary conditions are specified at only one value or at more than one value of the independent variable. This book, which is one of a series of monographs on numerical analysis, aims at a treatment of the latter class only. It deals largely with finite-difference methods of solution and the author states his intentions clearly in the preface. Thus: "Throughout I have taken the view that truncation errors in finite-difference equations should not be tolerated, and have made full use of the 'difference-correction' method to eliminate such error without decreasing the finite-difference interval."

The first three chapters assemble all the necessary basic information. Finite-difference equations and the technique of differencing are dealt with and the reader is quickly introduced to questions of accuracy and precision in numerical solutions. Chapter 3 deals with the numerical solution of algebraic equations by both the relaxation method and a more direct matrix elimination method. Both methods have their advantages in individual cases, and their merits are discussed and well illustrated by numerical examples.

The next four chapters deal with the usual classified types of differential equations. There are chapters on second and higher-order equations, both linear and non-linear. A separate chapter deals with first-order equations which, although essentially of initial-value type, are here treated by boundary-value methods. Eigenvalue problems are treated in detail but, apart from the use of the Rayleigh quotient, there is not a very full discussion of other methods of extracting eigenvalues from linear systems. Chapter 8 is interesting in that it describes the solution of boundary-value problems using initial-value techniques and, together with the chapter on first-order equations, which does the reverse, should prove instructive to the computer in the essential differences in the two techniques. Chapter 9 deals with the important question of accuracy of solutions and the final chapter deals with some further miscellaneous methods of solution. Here, amongst other things, Richardson's 'deferred approach to the limit' is discussed as an alternative to the 'difference correction' method. There is of course, in view of the author's stated intentions, little discussion of the relative merits of these methods compared with the use of truncated finite-difference formulae on smaller

intervals of tabulation. The briefest of mentions of methods independent of finite-differences, given in the last few pages of the book, scarcely does them justice.

The book is well presented and written in a straightforward style. References for further reading are adequate though not exhaustive. There are numerous worked examples, often solved more than once by alternative techniques. The unwary computer is repeatedly warned in a way that is scarcely possible in books covering a wider field of numerical analysis. Dr Fox has written an authoritative and readable book which can be highly recommended to all workers in this field.

S. C. R. DENNIS

**Introduction to Operations Research.** By C. WEST CHURCHMAN, R. L. ACKOFF and E. L. ARNOFF. 96s. 1957. (Chapman & Hall)

The editors, who with twelve other experts are responsible for the contents of this book, are members of the Operations Research Group in the Case Institute of Technology and thus the volume of well over 600 pages is representative of what is regarded, in centres of higher education, as the discipline of industrial Operational (in the U.S.A. Operations) Research.

It is well known that O.R. was born and bred in the military field, but after the end of World War II the transfer of interest to managerial problems in industry and business altered not only its substance, but also the level of executive authority at which it is aimed. Courses and seminars are now being held at Universities, Polytechnic Institutes, etc. on both sides of the Atlantic.

The introductory Part I gives a very clear account of "The General Nature of O.R.". It mentions as phases the formulation of a problem (dealt with, in more detail, in Part II), the construction of a mathematical model (Part III), its testing, and finally its implementation (Part IX). This establishes O.R. as a branch of scientific activity. The processes to be discussed are then stated as Inventory, Allocation, Waiting Line (i.e. Queuing), Replacement, and Competitive Processes. These are dealt with in Parts IV-VIII.

The editors emphasize that the working unit of O.R., as conceived by them, is a team rather than an individual, and this emerges also from Part X: "Administration of O.R.".

There are altogether 22 chapters spread over the ten parts. They contain some theory, and many examples. Chapter 2 consists of the description of a fictitious company which produces machine tools, and of the problems which might be put to its O.R. team. Mathematicians will perhaps be mainly interested in the theories on which O.R. workers draw. They are of a statistical nature, and some recent branches of applied mathematics have been developed with an O.R. application in mind. Of these, the authors deal with Linear Programming, Game Theory, Queuing Theory, Inventory and Renewal Theory. The treatment is rather elementary, but succeeds in explaining the essentials in a satisfactory way.

As to the practice of O.R., the reviewer doubts that this art can be learned from descriptions of examples which are often, for understandable reasons, purged of realism; they may be useful as a background, though. Even so, the book seems to fall between two stools. It is too superficial for those who want to understand, and perhaps to develop, the theory, while the practical man will find that what he needs is, above all, practice.

S. VAJDA

**Éléments de Mathématique XXI.** By N. BOURBAKI. Livre VI. **Intégration.** Chapitre 5. **Intégration des Mesures.** Actualités scientifiques et industrielles 1244, (Hermann, Paris)

This is a continuation of the Bourbaki series on integration; a knowledge of the first four chapters is presupposed. (See these reviews vol. 38 (1954), p. 230).

Let  $X$  be a locally compact space,  $K(X)$  the space of continuous functions on  $X$  with compact support,  $\mu$  an integral with domain  $K(X)$ . By standard procedures,  $\mu$  is extended to an integral on a class of functions called " $\mu$ -integrable" and thence, by disregarding "locally negligible" functions (that is, those whose upper integrals over every compact set vanish) to "essentially  $\mu$ -integrable" functions.

Now let  $T$  be a locally compact space with a non-negative measure  $\mu$ ; and let  $\lambda_t (t \in T)$  be a positive measure on  $X$  such that for each  $f \in K(X)$ ,  $t \rightarrow \int f d\lambda_t$  is essentially  $\mu$ -integrable, and  $t \rightarrow \lambda_t$  is vaguely  $\mu$ -measurable. Such a family of measures is called  $\mu$ -adequate. Now for suitable functions  $f$  on  $X$ , whose values may be real numbers or elements of a Banach space)  $\int d\mu(t) \int f(x) d\lambda_t(x)$  is defined. Denote this integral by  $v = \int \lambda_t d\mu(t)$ ; for such  $v$ -integrable  $f$  the function  $t \rightarrow \int f(x) d\lambda_t(x)$  is essentially  $\mu$ -integrable and  $\int f(x) dv(x) = \int d\mu \int f(x) d\lambda_t(x)$ . Now let  $\pi$  be an application of  $T$  in  $X$ , and let  $g$  be a finite positive function on  $T$ . The couple  $(\pi, g)$  is " $\mu$ -adapted" if  $\pi$  and  $g$  are  $\mu$ -measurable, and if for every  $f \in K(X)$  the application  $t \rightarrow f[\pi(t)]g(t)$  is essentially  $\mu$ -integrable. Let  $\varepsilon_x$  denote the unit measure in  $X$  concentrated at  $x$ . Then if  $(\pi, g)$  is  $\mu$ -adapted, the family of measures is  $t \rightarrow g(t)\varepsilon_{\pi(t)}$  is  $\mu$ -adequate so that  $v = \int g(t)\varepsilon_{\pi(t)} d\mu(t)$  is defined, and a function  $f$  (with values possible in a Banach space) is essentially  $v$ -integrable if and only if  $t \rightarrow f[\pi(t)]g(t)$  is essentially  $\mu$ -integrable, in which case  $\int f(x) dv(x) = \int f[\pi(t)]g(t) d\mu(t)$ . A measurable application  $\pi$  of  $T$  in  $X$  is " $\mu$ -proper" if  $\pi$  is  $\mu$ -measurable and if for every  $f \in K(X)$ ,  $f \circ \pi$  is essentially  $\mu$ -integrable. Then the image measure  $\int \varepsilon_{\pi(t)} d\mu(t)$  on  $X$  is defined; it is denoted by  $\pi(\mu)$ . Then if  $f$  is a suitable function on  $X$  (whose values may be in a Banach space),

$$\int f(x) d\pi(\mu) = \int f[\pi(t)] d\mu(t).$$

Applications mentioned in the body of the text include Fubini's theorem, the formula for change of variable in integrating functions of a real variable, the Radon-Nikodym theorem, and the duality of the spaces  $L^p$  and  $L^q$  for  $p, q \geq 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . There are many other applications in the numerous and interesting exercises.

Those who have read the previous chapters will probably find the mixture here very much as before; those who have not would be well advised to do so before embarking on this chapter. In the opinion of the reviewer, this work is not suitable for an introduction to the subject of measure and integration.

D. M. STONE

**Colloque d'Algèbre Supérieure Tenu à Bruxelles du 19 au 22 Décembre 1956.** Centre Belge de Recherches Mathématiques 1957. 293 pp. (Ets Ceuterick, 66 Rue Vital Coster, Louvain, Belgique. 250 Belgian francs)

This Colloquium, organised by the CBRM in 1956, consists of 12 lectures covering many aspects of modern algebra, particularly the applications to geometry and the 'algebraisation' of theories. The latter includes an axiomatic study, by M.-L. Dubreil-Jacotin, of the mean values introduced by Osborne Reynolds. P. Dubreil surveys the algebraic problems in the theory of semigroups, and L. Lesieur and R. Croisot sketch a generalisation of the theory of Noetherian rings for the noncommutative case. In general valuation theory W. Krull examines some questions arising in the non-Archimedean case; M. Krasner treats the approximation of complete valued fields of finite characteristic. Group theory is represented by G. Higman with a survey of the state in 1956 of the Burnside problem, and by J. A. Green who describes the characters of the general linear group  $GL_n(F_q)$ . J. Tits studies the Chevalley groups by means of geometries associated with the semisimple complex Lie groups. L. Lombardo-Radice discusses questions centring on configuration theorems in projective planes. Recent work on local algebra is summarised by P. Samuel. E. Witt has some remarks on quadratic forms. E. Waelbroeck studies the sub-algebra of regular elements in a topological algebra.

The lectures vary from the expository talk to the detailed paper with complete proofs, but they all abound in unsolved problems and thereby form a most stimulating whole.

P. M. COHN

**Electricity and Magnetism.** By B. I. BLEANEY and B. BLEANEY. Pp. xiv + 676. 63s. 1957. (Clarendon Press, Oxford)

Professor Bleaney and his wife have fitly signalled the appointment of the former to Dr Lee's Chair of Experimental Philosophy and to the headship of the Clarendon Laboratory by publishing this magnificent textbook on electricity and magnetism. So far as the reviewer is aware this is the only book in English which provides an up-to-date and comprehensive account of this subject for undergraduates and first-year graduates, with adequate mathematical apparatus, a rationalised treatment of the classical theory, a modern presentation of the impact

of quantum theory and a discussion of the applications to both light and heavy electrical engineering.

This great treatise (which weighs about  $3\frac{1}{2}$  lbs) falls naturally into three parts—Chapters I–X on classical electromagnetic theory, Chapters XI–XVI on applications to engineering, Chapters XVII–XXIII on molecular, atomic and nuclear problems. A final chapter discusses the subject of units with special reference to the m.k.s. system which is adopted in the book. Two appendices give the necessary theory of vectors and numerical values of the fundamental constants.

Although the theory of the first ten chapters is fitly described as “classical”, it is by no means an antiquarian study. The electrophorus and cat’s skin have gone to make way for the Van de Graaff electrostatic generator, electron optics and the cyclotron. These chapters give a rapid, concise and full account of the theory of electrostatics, steady currents, magnetostatics, electromagnetic induction, alternating current theory and Maxwell’s equations, with applications to practical measurements of magnetic properties, and direct current measurements.

The second section provides a novel and welcome addition to an undergraduate textbook on electricity. It contains a detailed account of filters and waveguides, a full discussion of thermionic vacuum tubes (three chapters), and chapters on alternating current measurements and electromagnetic machinery.

The last section deals with the “microscopic” as distinct from the “macroscopic” theory, and describes the impact of quantum theory on electromagnetism. It gives modern theories of the dielectric constant, of conduction in the solid state, of paramagnetism, ferromagnetism, and accounts of the theories of fluctuation and noise, and of magnetic resonance.

The special theory of relativity lies outside the scope of this book, and it is to be hoped that we may have another volume from the Clarendon Laboratory giving a systematic and up-to-date exposition of this subject, which would be developed more naturally perhaps from the optical and astronomical standpoints.

There is a good index and numerous references to recent experimental and theoretical work. The book is beautifully printed and produced and will take its place at once as the foremost textbook on electromagnetism.

G. TEMPLE

**Taschenbuch der Mathematik.** By I. BRONSTEIN and K. SEMEN-DJAGEW. Pp. 548. DM 22.50. 1958. (Teubner, Leipzig)

Translated from the 3rd Russian Edition by V. Ziegler this pocket book is a very well presented summary of mathematics for the technologists. There are extensive tables and lists of formulae, including over 500 indefinite integrals. Amongst the topics illustrated by worked examples are Fourier series, differential equations, calculus of variations and conformal transformations.

R. L. G.

**Elementary Number Theory.** By E. LANDAU. Pp. 256. \$4.95. 1958. (Chelsea, New York)

**Functions of Real and Complex Variables.** By W. F. OSGOOD. Pp. 407 + 262. \$4.95. 1958. (Chelsea, New York)

This translation of Landau's *Elementare Zahlentheorie* has been prepared by J. E. Goodman with added exercises by P. T. Bateman and E. E. Kohlbecker. Landau's wonderful gift of exposition has been safely preserved in the translation and no further recommendation is necessary. Amongst the topics treated in this introductory volume are Brun's sieve, decomposition in two, three and four squares and the class number of binary quadratic forms. This is not just a book but a living record of mathematical discovery.

The reprints of Osgood's two volumes on real variable and complex variable have been bound together, forming a comprehensive introduction to analysis which is wonderful value for \$4.95.

R. L. GOODSTEIN

**25 Nobel Preisträger.** Biographies by J. HAUSEN. (Friedrich Vieweg & Sohn. Brunswick)

The 25 Nobel Prizewinners are those whose works have been published by Vieweg & Son, and include Lorentz, J. J. Thompson, Rutherford, Einstein and Born. Brief bibliographies, with details of the field of study are enlivened by some excellent photographs.

R. L. G.

**The Blue and Brown Books.** By LUDWIG WITTGENSTEIN. Pp. 185. 25s. 1958. (Basil Blackwell, Oxford)

The so-called blue and brown books comprise notes dictated by Wittgenstein, the former to a class during 1933-34 and the latter to Francis Skinner and Alice Ambrose during the following year. This printing makes accessible to the general public notes which till now were the treasured possession of a fortunate few. Skinner's note books, however, contain material, with corrections in Wittgenstein's handwriting, which has not been included in this printing.

R. L. GOODSTEIN

**Grundlagen und Analytischer Aufbau der Geometrie.** By L. HEFFTER. Pp. 191. 30s. 6d. 1958. (Teubner, Stuttgart)

This book, now in its third edition, gives a thorough and well-written account of the foundations of projective geometry of one, two and three dimensions, and its specialisations to affine, Euclidean and the non-Euclidean geometries. The treatment is axiomatic at first, but co-ordinates are introduced later by the usual method of cross-ratios.

The nearest equivalent book in English is probably that of O'Hara and Ward.

E. J. F. PRIMROSE

**Functions of Complex Variables.** By P. Franklin. Pp. ix, 246. \$6.95. 1958. (Prentice-Hall, New York)

**Funktionentheorie.** By H. Kneser. Pp. 422. 1958. (Vandenhoeck & Ruprecht, Göttingen)

There are many good introductions to complex variable theory, with a fairly well defined common core, so that new books can hardly fill a gap or present much novelty. These two are clear and readable; the differences in form and content probably spring from a difference in purpose.

Franklin aims at a rigorous simplicity which shall present the theory in a form suitable for those who wish to use it in some other field. He does not scamp his proofs but does not strain after generality for its own sake: for instance, he is content to prove Cauchy's theorem under Goursat's conditions. The elementary functions are studied *ab initio*, and following immediately are two good chapters on simple conformal mappings and the bilinear transformation. This is, I am sure, the right order for the purpose of the book. We then come to the integral theorems, the Taylor and Laurent series, and the evaluation of integrals by the residue calculus, with a brief note on the Mittag-Leffler expansion. There is nothing about many-valued functions or special non-elementary functions. The book would be a good foundation for those who have to apply complex variable methods to hydrodynamics or elasticity.

Kneser's book is for those who are interested in the subject for its own sake, the young analysts in the making. Cauchy's theorem comes early and is the basis for a long account of one-valued functions; this is taken far enough to include the Mittag-Leffler expansion, Weierstrass's factor theorem, and Jensen's formula, illustrated by notes on the gamma function, the elliptic and elliptic modular functions and Riemann's  $\zeta$ -function. Many-valued functions are exhibited on the Riemann surface; algebraic functions and Abelian integrals are discussed and related to linear differential equations. Finally there is a long chapter on the theory of conformal mapping; there is little illustrative matter, the central point being Riemann's mapping theorem, with a detailed study of its implications and a brief introduction to the conformal mapping of multiply-connected domains.

T. A. A. B.

**Vektoren und Matrizen.** By SIEGFRIED VALENTINER. Pp. 202. DM 4.80. 1958. (Walter de Gruyter, Berlin)

This is another volume in the Sammlung Götschen, a series of small paper-backed books on various subjects.

The book is written primarily for applied mathematicians and engineers, and it is pleasing to see that the treatment of the theory is completely rigorous. After developing vector field theory the author gives examples of important theorems on potential, hydrodynamics, electricity and magnetism which are best treated by vector methods. Matrices follow naturally from the vector theory, and some examples of practical interest are worked out by their use. The book ends with a collection of interesting examples provided by Professor König.

The author and publisher have combined to produce a book of high quality very cheaply.

E. J. F. PRIMROSE

**Toeplitz Forms and their Applications.** By ULF GRENANDER and GABOR SZEGÖ. Pp. vii, 245, \$6.00. 1958. California Monographs in Mathematical Science, 2, (University of California Press)

Toeplitz forms are the bilinear forms generated by a Hermitian matrix  $(c_{m-n})$ ; they are closely associated with the integral operators generated by symmetric function of the form  $k(x-y)$ . They are important in a number of problems of analysis, notably the moment problem, the theory of orthogonal polynomial and of power series, and in probability theory in, for example, the theory of stationary stochastic processes. The two authors, eminent respectively in statistical research and analysis, give a well written and comprehensive account of these matters.

J. L. B. COOPER

**Kontinuierliche Geometrien.** By FUMITOMO MAEDA. Pp. 244. DM 39. 1958. (Springer, Berlin)

A continuous geometry is, roughly speaking, a projective geometry in which the dimensions of the linear subspaces are the real numbers between 0 and 1. Its precise formulation however is in terms of the lattice of subspaces, and the whole subject is part of lattice theory. It was created in 1936-37 by von Neumann, but his publications were brief and a complete exposition was only available in the form of Princeton lecture notes. Subsequent work in this field has been predominantly Japanese and this translation of Professor Maeda's book is probably the first treatise on the subject printed in a western European language.

The first few chapters develop elementary lattice theory, and lead on to a study of the lattice of subspaces of a projective space. Then follow two chapters on the special lattices related to continuous geometries (continuous complemented modular lattices). The main result here is that it is possible to define a unique dimension function on such lattices.

In an ordinary projective geometry of dimension greater than two it is well known that one can introduce coordinates from a (possibly skew) field. Translated into lattice theory, and suitably generalized, this leads to the following problem: can one represent a continuous geometry as the lattice of principal right ideals of a suitable matrix ring? The second half of this book is devoted to the solution of this problem.

M. F. ATIYAH

**Homology Theory on Algebraic Varieties.** By A. H. WALLACE. Pp. 115. 32s. 6d. 1958. (Pergamon Press, London)

In 1926 Lefschetz published his fundamental work on the topology of algebraic varieties. In the intervening years it has come to be recognized that, like so much pioneer work, Lefschetz's proofs relied too much on intuition, and various attempts have therefore been made to re-establish the theory on solid foundations. Dr. Wallace's book is one such attempt, and there is little doubt that it will stand up to the

most rigorous criticism. The proofs which he gives are, in his own words, "fairly elaborate and involve a considerable amount of verification of intricate detail".

There is one respect however in which this work is deficient, for whereas Lefschetz's theory includes properties of the fundamental group the theorems considered by Dr. Wallace are, as indicated in the title, restricted to homology. It is therefore of considerable interest that a new approach to the subject has recently been discovered by Thom and Bott in which the whole Lefschetz theory can be rigorously deduced in a couple of pages from standard Morse theory.

M. F. ATIYAH

**Some Properties of Differentiable Varieties and Transformations.** By B. SEGRE. Pp. 183. DM 36. 1957. (Springer, Berlin)

The diversity of the subject matter of this book is partly balanced by a certain uniformity of treatment. Essentially this can be put down to the fact that this is a book on differential geometry written by an algebraic geometer, and the ideas and concepts of algebraic geometry are dominant throughout. Despite this the book still seems to lack any positive theme, and is consequently difficult to digest. There appear to be a large number of new results, but their importance is not easy to assess. Perhaps the most important section, and the one with the widest appeal is that dealing with local analytic transformations (Chapter II). This is a difficult topic with some really serious obstacles, and the author claims to have made new contributions to it.

M. F. ATIYAH

**A Survey of Binary Systems.** By R. H. BRUCK. Pp. 185. DM 36. 1958. (Springer, Berlin)

It is probably true that both in modern algebra and its applications the most important concept is that of a group. Nevertheless a generalisation of this concept is often needed, and there exists today a wide variety of these 'almost-groups'. Prof. Bruck is concerned with giving a survey of those algebraic systems which admit a single binary operation, usually but not always single-valued. Even with this limitation a report aimed to be exhaustive would not be feasible; the author has therefore devoted the main part of his book (about two thirds) to a detailed report, with full proofs, on the theory of loops. The topics discussed, many of them taken from the author's papers, are homomorphism theory, Lagrange's theorem, nilpotency and Moufang loops. A loop is Moufang if it satisfies the identity  $(xy)(zx) = (x(yz))x$ , and a considerable amount of structure theory is developed for such loops. In particular, the theorem—conjectured by T. Slaby and proved by him for  $n = 2$  and  $3$ —that every commutative Moufang loop on  $n$  generators is centrally nilpotent of class at most  $n - 1$ , is proved here for the first time.

The first third of the book gives a concise survey of the different binary systems. It is not even possible to mention all the systems in

this review, since there is no complete agreement on nomenclature, but the author has managed to include at least the definition and some basic properties of most of the common systems. Rather than cram his pages with results, he has been content to add a few theorems, mostly with proofs, chosen for their interest, giving this section more the character of an anthology. Semigroups receive the most systematic treatment, about twenty pages being devoted to their properties, especially the ideal theory. For more details the reader is often referred to the bibliography, covering some 450 items. The subdivision of the bibliography into the different subjects is helpful, as well as the page references to Mathematical Reviews, but it is unfortunate that the rather lengthy supplement could not be incorporated in the body of the bibliography.

The book is written with an enthusiasm which communicates itself to the reader and, one feels, serves him better than a really complete report. At the same time one hopes that it will encourage someone to do for semigroups what Prof. Bruck has done for loops.

P. M. COHN

**Polynomial Expansions of Analytic Functions.** By R. P. BOAS and R. C. BUCK. Pp. 77. DM 19, 80. 1958. (Springer, Berlin)

According to the preface the authors have tried to bring about a certain amount of order and completeness and to formulate results and methods in a fashion which will make them more generally accessible. They have succeeded in producing a very compact and readable account of a great deal of recent work and its more classical background. There is a twenty page introduction which reaches a form of generalised Appell polynomials. Proceeding to more special cases they discuss properties and expansion theories of the polynomials of Bernoulli, Laguerre, Hermite, Tehebycheff, Jacobi and many others. A brief but suggestive final chapter on applications to uniqueness problems and functional equations must be mentioned and the authors commended for their provision of a detailed contents list, index, bibliography, and list of special symbols.

A. J. MACINTYRE

**Univalent Functions and Conformal Mapping.** By J. A. JENKINS. Pp. 177. DM 34. 1958. (Springer, Berlin)

A short but very scholarly account of the known results of the theory with detailed references but not proofs is followed by three chapters covering nearly sixty pages and culminating in the proof of the author's 'general coefficient theorem'. Any reader who has battled his way through this will be richly rewarded by applications to existence theorems for conformal mapping (stopping short of Koebe's  $k$ -circles theorem) and many if not most of the known solutions of extremal problems for univalent functions in simply and multiply connected

domains, involving the domains of variability of  $f(z)$ ,  $f'(z)$  etc, for fixed  $z$  as well as more elementary results. Finally in the last chapter the author combines his method with symmetrisation to prove results for  $p$ -valent functions which go well beyond anything that could be obtained in any other way.

The bibliography of nearly two hundred papers is excellent and together with the authors' references enables the reader to find easily who was 'really' the first person to prove any given result. This leads to some surprises. It is perhaps a pity that the author was determined not to use any method except that of the extremal metric pioneered by himself. Consequently some simple and fundamental theorems such as Littlewood's  $|a_n| < en$  are not proved at all and the proofs of others are unnecessarily complicated.

W. K. HAYMAN

**Darstellende Geometrie I.** By W. HAACK. Pp. 113. DM 12.40. 1958. Göschen Collection 142. (de Gruyter, Berlin).

**Darstellende Geometrie.** By F. REHBOCK. Pp. 232. DM 26.80. 1957. (Springer, Berlin).

The first of these books covers adequately the usual elementary work associated with the representation of points, lines and planes, the sections of parallelepipeds and pyramids: no mention is made of curved surfaces. The diagrams are very small, but the explanatory text is clear and precise.

The second book covers a much wider field including, for example, shadows and twisted curves: the latter half of it deals with the principles and applications of the principles of perspective. The book is biased towards the requirements of Civil Engineering and seems to have been written for students in that faculty of the Technische Hochschule in Brunswick. The text, written in simple straightforward language, is placed on each of the left-hand pages and the associated diagrams, somewhat larger than usual in such books, are placed on the right-hand pages: this, together with the decimal subdivision of the various sections makes for ease of reference. The book is worthy of a place in any technical library used by engineers and mathematicians.

F. T. CHAFFER

**Il Teorema di Riemann-Roch per Curve, Superficie e Varietà: Questioni Collegate.** By F. SEVERI. Pp. viii, 131. DM 22.60. 1958. (Springer, Berlin)

Modern generalisations of the Riemann-Roch theorem may tend to obscure its humble origin in the theory of linear series of sets of points on an algebraic curve, and its classical extensions at the hands of the Italian geometers to linear systems of curves on a surface, and of hypersurfaces on a variety. Professor Severi, in this book, has given an account of this algebro-geometric theory, to which he has himself made outstanding contributions, and has somewhat extended its scope

in that he deals with problems of a similar character connected with algebraic rather than linear equivalence, and with his own theory of series of equivalence of sets of points on a surface. As a result we have what is, in effect, an outline of a considerable part of the history of algebraic geometry, viewed from the algebro-geometric standpoint. In a work of this length the proofs of the theorems are, of necessity, not given in complete detail (though very complete references are given); in consequence it is much easier to get a general view of the subject, and to see the precise difficulties which arise in trying to extend the simple theory of curves to surfaces and varieties of higher dimension.

J. A. TODD

**Praktische Mathematik.** By R. ZURMÜHL. Pp. 524. DM 28.50. 1957. (Springer, Berlin)

This book, which deals with numerical methods, is intended to supplement and extend the basic instruction in Mathematics given to students of Physics and Engineering in the Technische Hochschulen. The treatment of the subject is sufficiently rigorous to make the book interesting to the pure mathematician since the author adopts the principle that, while the technologist is primarily interested in the practical application of a mathematical concept, he is all the more likely to apply a method correctly or to choose the best from many methods if he has first understood the mathematical groundwork of the process to be used.

The book starts with the definition of a complex number as an ordered pair and then proceeds to deal with methods for finding roots of equations, including the iteration process and the methods of Newton, Horner and Graeffe, leading on to the root stability criteria of Routh and Hurwitz. This is followed by a lucid account of matrix algebra, and very considerable use is made of this topic in the rest of the book in dealing with such problems as the solution of systems of equations, distribution of errors and boundary and eigenvalue problems associated with ordinary differential equations. Other topics dealt with include interpolation by the methods by Gauss, Gregory-Newton, Lagrange, Laplace and Stirling, numerical integration, the use of normal equations (but not the method of correlates), dispersion, the  $t$  and  $\chi^2$  tests and significance, harmonic analysis and curve fitting.

The text is illustrated by numerous worked examples; these are of the type that may well occur in practice and have obviously been chosen for the reason that they illustrate a method and not merely because they produce a simple or compact result; in fact the whole work seems to have been written on the assumption that a desk calculator is its natural concomitant. The book is written in a straightforward style, well printed on good paper and may be recommended to all who have to use or teach numerical methods.

F. T. CHAPTER

**Functional Analysis and Semi-groups.** (Revised Edition) By EINAR HILLE and RALPH S. PHILLIPS. Pp. xii, 808. \$13.80, 1957. American Mathematical Society Colloquium Publications, Vol 31. (American Mathematical Society)

The first edition of this book has been for ten years the leading work in the English language on analysis in Banach spaces and algebras as well as on the theory of semi-groups and the innumerable topics which can be grouped under that name. The second edition has been largely rewritten, with the collaboration of the second author, and much new material bearing on the more recent research work in the very active field of its subject has been added. In addition the expositions of the fundamentals of the theory of topological vector spaces, Banach spaces and Banach algebras have been simplified and expanded, making this a good introduction to these subjects. The exposition throughout is clear, and the book is assured of its place as an indispensable classic.

J. L. B. COOPER

**Commutative Algebra, Vol. I.** By O. ZARISKI and P. SAMUEL. Pp. xi, 329. 52s. 6d. 1958. (D. van Nostrand, Princeton)

The theory of Noether rings has been studied intensively for about twenty years, receiving attention both from algebraists and from geometers needing it as a tool. The outcome of these varied efforts is a remarkably deep and fascinating part of algebra. In addition, during the last decade, there has been increasing interest from algebraists who are concerned with the possibility of generalising the theory to non-commutative rings, that is, rings with maximum condition on one-sided ideals. In view of all these developments it is surprising that very few accounts of the theory have appeared, only the *Ergebnisse* monograph of W. Krull and the Cambridge tract of D. G. Northcott, both necessarily very brief, come readily to mind. The present work, of which this is the first volume, sets out to rectify this omission and to provide a thorough account of Noether ring theory. It performs this difficult task with great success.

The first volume begins with a considerable introduction to commutative rings, modules and fields (mainly the theory of algebraic extensions), followed by chapters on general Noether rings and classical ideal theory (Dedekind domains). Local rings are essentially left over for the second volume. The presentation is good throughout and there are some useful bonuses, notably the remarks which link the Noether theory with the structure theory of rings with minimum condition and the sketching of alternative methods of proof of important theorems.

The book is a pleasure to read and its many valuable features will ensure its use both to the specialist and to the beginner needing an introductory text in this field.

A. W. GOLDIE

## BRIEF MENTION

**A University Algebra.** By D. E. LITTLEWOOD. Pp. 324. 30s. 2nd Ed. 1958. (Heinemann)

In this new edition, the chapter on the laws of algebra has been rewritten, and a more detailed account of the classical theory of ideals has been added.

**Mathematical Tables.** By H. B. DWIGHT. Pp. 217. \$1.75. 1958. (Dover, New York. Constable, London)

This Dover Edition is similar to the first edition published by McGraw-Hill, except that tables in hundredths of degrees have been omitted, and a number of tables have been added, including some types of Bessel functions.

**Microwave Transmission Design Data.** By T. MORENO. Pp. 248. \$1.50. 1958. (Dover, New York. Constable, London)

An unaltered and unabridged republication of the first edition of a reference handbook for radio engineers.

**Calculus of Variations.** Vol. VIII. Proceedings of the Eighth Symposium in Applied Mathematics. Pp. 153. 58s. 1958. (McGraw-Hill)

The applications discussed relate to elasticity and plasticity, hydrodynamics, dynamic programming, and conformal mapping.

**Mathematics for Engineers.** Part II. By W. N. ROSE. Vth Edition. Pp. 403. 25s. 1958. (Chapman & Hall)

In addition to ten chapters on the Calculus there is a chapter on spherical trigonometry and one on probability.

**Integrals of Airy Functions.** U.S. Department of Commerce. National Bureau of Standards. Applied Mathematics Series 52. Pp. 28. 25 cents. 1958. (Washington, D.C.)

The functions tabulated are

$$f(x) = \int_0^x Ai(-t) dt, \quad F(x) = \int_0^x f(t) dt$$

where

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

for  $x$  from  $-2.00$  to  $5$ , and the modified Airy integral

$$A_0(x) = \int_0^\infty e^{-xt - t^3} dt$$

for values of  $x$  from  $0.00$  to  $10.00$ .

**Tables of Transport Integrals.** By W. M. ROGERS and R. L. POWELL. Pp. 46. 40 cents. 1958. (U.S. Government Printing Office, Washington 25, D.C.)

This table is the National Bureau of Standards Circular 595, and gives the values of the integrals

$$J(x) = \int_0^x \frac{e^z z^n}{(e^z - 1)^2} dz$$

to six significant figures for integral values of  $n$  from 2 to 17 and for values of  $x$ , by intervals of 0.1, from 0.1 to 25 ( $n = 2$ ) and to 40 ( $n = 17$ ).

**Vierstellige Tafeln zum Praktischen Rechnen in Unterricht und Beruf.** By PH. LÖTZBEYER. Pp. 44. 1958. (W. de Gruyter, Berlin)

This table book is printed in a very clear bold face and contains, in addition to the material of the familiar school tables an account of nomograms and a long list of formulae.

**Table of Natural Logarithms for Arguments between Five and Ten to Sixteen Decimal Places.** Pp. 506. \$4.00. 1958. (U.S. Government Printing Office, Washington D.C.)

This table is number 53 in the National Bureau of Standards Applied Mathematics Series, and supersedes Mathematical Table 12. The table contains 16-place values of the natural logarithms from 5 to 10 at intervals of 0.0001.

**Anfangswert Probleme bei Partiellen Differential Gleichungen.** 2nd Ed. By R. SAUER. Pp. 284. DM 41. 1958. (Springer, Berlin)

The principal change in this second edition is the addition of a Chapter on Schwartz's theory of distributions.

**Magnetohydrodynamics.** Edited by R. K. M. LANDSHOFF. Pp. 115. 32s. 1957. (Stanford University Press & Oxford University Press)

This is a study of how magnetic fields influence and are influenced by ionized gases. The contributors include F. Hoyle, on the build-up of large magnetic fields inside stars and J. M. Burgers on the penetration of a shock wave into a magnetic field.

### THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. M. A. Porter. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**

## REPORT OF THE COUNCIL FOR THE YEAR 1958

### *Membership*

During the year ended 31 October 1958, 289 ordinary members and 52 junior members were admitted to the Association. At the end of the year the membership figures were: Honorary, 10; Ordinary, 2,928; Junior, 123; Life, 259; a total of 3,320 compared with 3,084 at the beginning of the year.

It is with regret that the Council reports the death of the following members: Mr. J. B. Bretherton (1924), Mrs. V. T. Brownless (1949), Mr. R. H. Dick (1922), Mr. A. I. Lane (1956), Mr. B. J. Lewsley (1925), Mr. B. J. Martin (1951), Mr. G. L. Parsons (1930), (Honorary Secretary 1936-48, President 1955), Miss F. M. A. Pendry (1936), Dr. S. Rushton (1944), Mr. P. J. Smith (1936), Mr. J. F. Swain (1946), Mr. J. Tomlinson (1908).

### *Finance*

The Treasurer's statement for the year ended 31 October 1958, shows an excess of income over expenditure of £1,380. 0s. 8d. Included in the total income of £7,010. 13s. 1d. are the first grant of £1,000 from the Gulbenkian Foundation and a further £61 in donations from members. Altogether 78 members gave a total of £477, and 56 members made loans amounting to £1,459, in response to the appeal made by the President in 1957.

Prominent in the total expenditure of £5,630. 12s. 5d. is the sum of £1,571. 18s. 9d. which was spent on printing and reprinting the Reports. In spite of this, and £2,927. 0s. 1d. which was spent on printing and distributing the *Gazette*, the Association is in a position to meet its immediate liabilities without touching the War Loan Reserve. The whole amount of the loans from members will be repaid during January 1959, and the costs of publishing the last of the post-war series of major reports, that on the Modern Schools, can be met from the bank deposit account and the second grant of £1,000 from the Gulbenkian Foundation.

The Association has been approved by the Commissioners of Inland Revenue for the purpose of Section 16, Finance Act, 1958, and the whole of the annual subscription paid by a member who qualifies for relief under that Section will be allowable as a deduction from his emoluments assessable to income tax under Schedule E. Further details will be printed in *Gazette* number 343 for February 1959.

No decision has yet been reached about the future of the seven-year covenant scheme. The Association is to appeal before the Special Commissioners on 3 April 1959 against the decision of the Commissioners to stop repayment of income tax on covenanted

subscriptions from 1956 onwards. In the meantime it is important to note that, while members are bound by their seven-year covenants to the Association, they cannot take advantage of the benefit set out in the previous paragraph.

### *The Mathematical Gazette*

There has been a welcome increase in the number of articles of interest to schools which have been submitted for publication. Several new features have been started and the editors look to members for their continued support to maintain the new sections and to make further innovations possible.

### *Library*

The number of periodicals received in exchange for the *Gazette* has continued to grow and makes the Association Library one of the largest collections of mathematical periodicals in the provinces. Good progress has been made in clearing off arrears in binding. Library correspondence has increased but the number of volumes borrowed by members living outside Leicester remains small. Accessions during the year have been confined to gifts. All volumes listed in the new review section of the *Gazette* under the heading "Brief Mention" are placed in the Library.

The Librarian would be glad to receive copies of *Gazettes* Vols. I-III to help build up a new complete set.

### *The Teaching Committee*

The Teaching Committee appointed for the four-year period 1958-62 included ten new members, most of whom have been assigned to sub-committees. Sub-committees have been formed to revise the Mechanics Report and the Arithmetic Report; pending issue of revised reports the original reports have again been reprinted. A new sub-committee is considering the teaching of Statistics in schools. The work of the Modern Schools Sub-committee has been completed for the time being with the publication of their Report.

The Committee has also dealt with a reference from the Secondary School Examinations Council about sixth form courses for non-specialists, and has been represented in discussions about a proposed Diploma in Mathematics.

### *The Branches*

The home branches of the Association, while showing much diversity of practice and membership, all seem to have enjoyed successful and well-attended meetings during the past year.

Contacts have been maintained with overseas branches and it is pleasant to note the continuing evidence of stability and vigour in their annual reports.

*Problem Bureau*

Requests for solutions have been more numerous than in any other recent year, especially in the weeks following the appearance of the last report in the *Gazette*. Most of the requests have been for solutions to Cambridge Open Scholarship questions. Inquiries for the solution of "puzzles" have reappeared, after being scarce for some years. It must be emphasized that the services of the Bureau are open to any member of the Association at the cost of a stamped addressed envelope, and more requests can be handled quite easily. Requests for solutions to Cambridge Open Scholarship papers in *Mathematics* need only quote the numbers of volume, page, and question, e.g. 280/47/1. In all other cases a copy of the question must be sent. There has been some response to the suggestion in the last report, and file solutions will be available from this year for the Oxford and Cambridge Joint Board, and London G.C.E. "A" and "S" levels. Any one who would care to help by providing papers and solutions to a public examination of a similar standard would be assisting the Association very materially. All correspondence should be addressed to Dr. G. A. Garreau, 90 Wyatt Park Road, London, S.W.2.

*Officers and Council*

Once again the year has been one of extreme activity in the affairs of the Association. Twelve months ago it was reported that the membership had grown by over 300 to a record figure in excess of 3,000. This year the total is higher by a further 236. There has been much work done in committees, in the representation of the views of the Association in various ways, in issue of advice and information, and in preparation for the proposed Diploma in Mathematics.

The thanks of the Association should be recorded to all who have helped to further its aims; and in particular to the President and officers for their work on behalf of members generally.

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## ANNUAL GENERAL MEETING 1959

The Annual General Meeting of the Mathematical Association was held in Southampton from Wednesday April 8th until Saturday April 11th. The meeting was well attended and highly successful. The proceedings were initiated by a Reception and Tea in the Guildhall, an innovation which gave members an opportunity to admire Southampton's fine modern Civic Centre.

The Presidential Address on Thursday morning was given by Professor M. H. A. Newman F.R.S., on the subject "What is mathematics? New answers to old questions." Professor Newman's theme was a defence of postulational methods in the foundations of mathematics. This was followed by an account of the work of the Southampton Port Operation and Information Service ably given by the Harbour Master Captain J. Andrew M.B.E., and Mr. A. L. P. Milwright of the Radar Research Group at the Admiralty Signal and Radar Establishment. In the afternoon members divided into various groups which severally visited Stonehenge, the Docks and Winchester Cathedral and College, and other points of interest in the area.

On Friday morning Professor Richards, Head of the Department of Aeronautical Engineering spoke of the increasing use of mathematicians in the Aircraft industry, and this was followed by a discussion of the Associations' important new Report on the Teaching of Mathematics in Secondary Modern Schools, which unfortunately did not lead to the spirited debate which the discussion of Reports has so frequently provoked in the past. In the afternoon Dr. G. N. Lance described and demonstrated the University Pegasus Computer and Dr. J. P. Cave spoke of the possible future uses of Computers in translating languages. The final lecture of the day, by Mr. A. R. Pargeter of Taunton's School, was on plaited polyhedra, a brilliant account of some original work. The meeting closed with a lecture on Saturday morning on The Mechanics of the Fun Fair, by Professor Emeritus Wing Commander T. R. Cave-Brown-Cave, C.B.E., who followed an account of fair-ground machinery by an astonishing demonstration of the right way to lift heavy weights.

On Friday evening the Vice-Chancellor of Southampton University, Dr. D. G. James was the guest of the Association at dinner, and his graceful tributes to mathematics and mathematicians delighted us all. The menu was a complex one, the real part of which was served and eaten with evident pleasure; the imaginary part (created it was rumoured by Mr. Pargeter) deserves the more permanent record of a place in this Gazette.

A LA DESCARTES  
(Imaginary)

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Extract of soluble roots

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Poisson à la mode  
Eureka Source  
Compute Potatoes  
Les petits pois au Jordan

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Choice of { Segment of Abel Pi  
                  and homogeneous custard  
                  Pflaumen mit Einstein  
                  Polar Ice

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Parmesan Prisms epi Polygons

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Instant t  
n sugar<sup>3</sup>

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(Served on log tables to 200 significant figures)

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**"The Queen of the Sciences"**

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**"This Function"**

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*Proposition*  
*Converse*

PYTHAGORAS  
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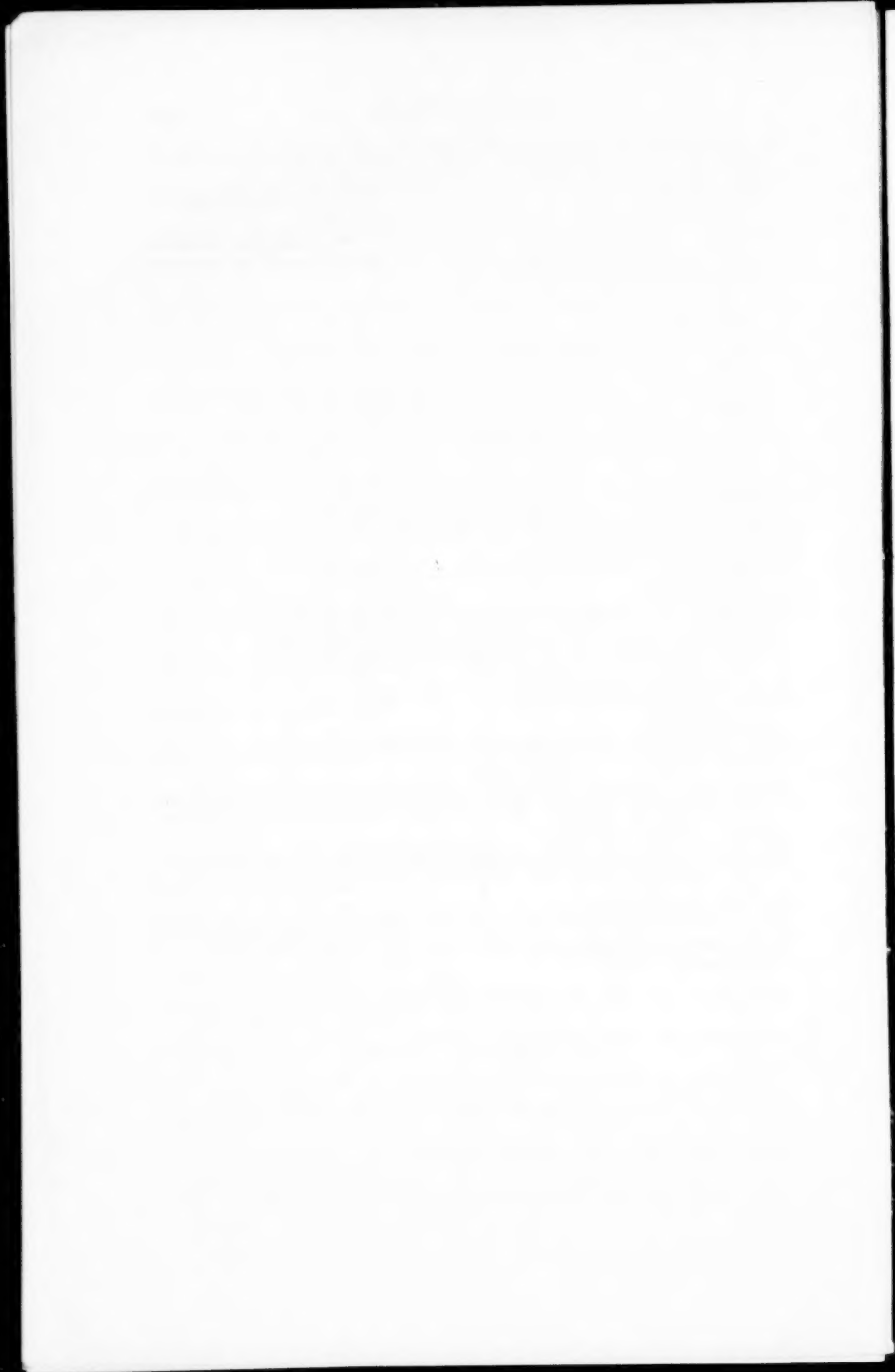
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